# Recall of First-Order Logic

## Exercises

## 1 Syntax of a first-order language

Consider the following alphabet:

- variable symbols: $x$, $y$, $z$
- function symbols: $a/0$, $b/0$, $c/0$, $d/0$, $e/0$
- predicate symbols: $p_f/1$, $p_m/1$, $q_m/2$, $=/2$, $p_r/3$

*Exercise 1.* Formalize the following sentences by means of first-order logic:

1. $a$ is $p_m$
2. $c$ is $p_f$ or $d$ is $q_m$ with $b$
3. $c$ is $p_f$ or $d$ is $q_m$ with someone between $b$ and $e$
4. $a$, $b$, $c$ are related by $p_r$ in at least one way where $a$ appears before $c$ in the triple
5. if $b$ is $p_f$, then someone between $a$ and $b$ is both $p_f$ and $p_m$
6. $c$ is $q_m$ with $e$, in both *directions* (i.e., $c$ is $q_m$ with $e$, and $e$ is $q_m$ with $c$)
7. $a$, $b$ and $e$ are all different
8. both $a$ and $b$ are $q_m$ with someone, and $a$ comes always first in the pair (i.e., that $a$ is $q_m$ with someone, not that someone is $q_m$ with $a$)
9. all predicates with arity greater than one are commutative (Note: as it is, this sentence does not represent a first-order idea!)
10. someone is $p_f$ but not $q_m$ with anyone (in any direction)
11. any two equal individuals are $p_r$ with at least another one (which will be the third element of triple) which is not equal to them
12. there is only one individual which is both $p_f$ and $p_m$

*Exercise 2.* Translate the following formulæ into words (there are many possibilities!):

1. $q_m(a, d)$
2. $p_f(a)$
3. $p_r(a, b, c) \vee p_r(b, a, c)$
4. $p_f(a) \vee p_f(b)$
5. $q_m(a, b) \to p_m(a) \wedge p_m(b)$
6. $\forall xy(q_m(x, y) \to p_m(x) \wedge p_m(y))$
7. $q_m(x, y) \to p_m(x) \wedge p_m(y)$
8. $\forall x(a = x)$
9. $\exists x p_f(x)$
10. $\exists x p_f(x) \vee \forall y(\neg p_f(y))$
11. $\forall x(p_f(x) \vee p_m(a))$
12. $\neg\exists y(\forall x p_r(x, c, y))$
13. $\exists xy(q_m(x, x) \wedge q_m(y, y) \wedge (\forall z(q_m(z, z) \to z = x \vee z = y)))$
14. $\forall x \forall x' \forall x''(= (x, x') \wedge = (x', x'') \to x'' = x)$

## 2 Semantics of a first-order language

Domain $D = \{\text{Jim Henle, Tom Tymoczko, Aristotle, Adrienne Rich, Madonna}\}$

- $a$ refers to Jim Henle (a mathematical logician)
- $b$ refers to Madonna (a singer $\wedge$ a dancer but $\neg$ a great actress)
- $c$ refers to Tom Tymoczko (a philosophical logician)
- $d$ refers to Aristotle (a philosopher $\wedge$ a scientist $\wedge$ a logician)
- $e$ refers to Adrienne Rich (a poet)
- $p_f(x)$ means that $x$ is a female
- $p_m(x)$ means that $x$ is a male
- $q_m(x, y)$ means that $x$ is married to $y$
- $p_r(x, y, z)$ means that $x$ and $y$ are parents of $z$

*Exercise 3.* Define new predicates to formalize the properties about these individuals (that someone is a dancer, a poet, etc.).

*Exercise 4.* Using also the predicates defined above, formalize that

1. Madonna is married to Aristotle
2. Aristotle is male
3. Aristotle is married to Madonna
4. Jim Henle is Madonna
5. Madonna is herself
6. everyone is herself (himself)
7. $x$ is a child of Aristotle and Tom Tymoczko
8. Adrienne Rich is not herself
9. Jim Henle is male and Tom Tymoczko is female
10. Tom Tymoczko is not married to Jim Henle
11. Tom Tymoczko is not married to anyone
12. there is a logician, which is a philosopher but is not Aristotle, which is married to a poet
13. if Madonna were a philosopher, then she would be married to a logician
14. all scientists are logicians
15. all scientists are logicians, apart from men

Only after building the formulæ, say whether they are true or false in the interpretation above (and assuming the knowledge of the real world)

*Exercise 5.* For each of the furmulæ you built in Exercise 4, find, when possible, another domain and another interpretation of constants and predicates (as simple as possible) such that the truth value of the formula changes (from **t** to **f**, and from **f** to **t**)

*Exercise 6.*

1. does 5 belong to this set?

$$X = \{n' \in \mathbf{N} \setminus \{n'' \mid n'' \leq 4\} \mid n' \text{ is greater than 7 whenever it is even}\}$$

2. what about 6?
3. and 8?
4. how would you write $p_X(m) =$ "$m$ belongs to $X$" as a logical formula?
5. why don't we specify from which set we pick $n''$ in $\{n'' \mid n'' \leq 4\}$?

## 3 Logical consequence

*Exercise 7.* Check the correctness of this logical deduction by considering all possible interpretations:

$$\{p \vee (q \wedge r),\ p \rightarrow \neg q,\ \neg p \rightarrow \neg r\} \models p$$

What about the domain of the interpretations? Why don't we need to mention it?

## 4 Syntax vs. semantics

*Exercise 8.* Try to find a proof for the deduction of Exercise 7, that is, try to prove $p$ given the premises $p \vee (q \wedge r)$, $p \rightarrow \neg q$ and $\neg p \rightarrow \neg r$.

- Note: this can be very tricky if you are not familiar with this stuff!
- Hint (1): apart from Modus Ponens, De Morgan laws, commutativity, associativity, distributivity, and the definition of $\rightarrow$, consider the following inference rule:

$$\begin{array}{lcr} \text{if we have} & F_1 \vee F_2 \ \text{ and } \ F_1 \rightarrow G_1 \ \text{ and } \ F_2 \rightarrow G_2 & \text{(premises)} \\ \hline \text{we can infer} & G_1 \vee G_2 & \text{(conclusion)} \end{array}$$

- Hint (2): use the (valid) logical axiom $p \vee \neg p$ as an additional premise