Standardization of Formulæ

Exercises

1 Skolem normal form

Exercise 1. Prove that the following equivalence rules are indeed valid formulæ:

$$
\begin{aligned}
(\forall x F \to G) &\leftrightarrow \exists x(F \to G) \\
(\exists x F \to G) &\leftrightarrow \forall x(F \to G) \\
(F \to \forall x G) &\leftrightarrow \forall x(F \to G) \\
(F \to \exists x G) &\leftrightarrow \exists x(F \to G)
\end{aligned}
$$

Hint: prove each direction by supposing it not to hold and deriving a contradiction.

Exercise 2. Find a proof for the Lemma on the existence of the prenex form: *The prenex form of a formula always exists.*

Hint: consider that finding a prenex form consists of applying a set of equivalence rules, in a certain direction...

Exercise 3. Is there any case where the prenex form is not unique, in a non-trivial way? That is, can two prenex forms of the same formulæ be different apart from the names of bounded variables?

Also think about whether the prenex form of F be defined as

- a formula which is in prenex form and is equivalent to F; or
- a formula which is in prenex form and is obtained from F by applying a certain set of equivalence rules

Exercise 4. Find a proof for the Lemma on the existence of the conjunctive normal form: *The conjunctive normal form of a formula always exists.*

Hint: similar to Exercise 2.

Exercise 5. Consider the formula

$$\neg((F_1 \wedge G_1) \vee (F_2 \wedge G_2))$$

By applying the rules for getting a conjunctive normal form, it is possible to get

$$(\neg F_1 \vee \neg G_1) \wedge (\neg F_2 \vee \neg G_2)$$

but also a much more complicated one (guess how?), which happens to be more restrictive than the first one: it takes the form

$$... \wedge (\neg F_1 \vee \neg G_1) \wedge (\neg F_2 \vee \neg G_2) \wedge ...$$

– why is it more restrictive (look at the structure)?
– how is it possible? Shouldn't they be equivalent (they are both equivalent to the original formula)? Where is the trick?

Exercise 6. Compute the Skolem normal form of the following formulæ:

1. $\exists x \forall y \forall z \exists u \forall v \exists w p(x, y, z, u, v, w)$
2. $\forall x \exists y \exists z ((\neg p(x, y) \wedge q(x, z)) \vee r(x, y, z))$
3. $\neg(\forall x p(x) \rightarrow \exists y \forall z q(y, z))$
4. $\forall x ((\neg e(x, 0) \rightarrow (\exists y (e(y, g(x)) \wedge \forall z (e(z, g(x)) \rightarrow e(y, z))))))$
5. $\neg(\forall x p(x) \rightarrow \exists y p(y))$