Standardization of Interpretations

Exercises

1 Herbrand Universe

Exercise 1. Is the Herbrand Universe of a formula finite? Countable? Uncountable? Prove your guess.

Exercise 2. What is the necessary and sufficient condition for $H(F)$ to be finite?

2 Herbrand Base

Exercise 3. Is the Herbrand Base of a formula finite? Countable? Uncountable? Prove your guess.

What is the cardinality (i.e., the number of elements) of the Herbrand Base w.r.t. the Herbrand Universe?

3 Herbrand Interpretations

Exercise 4. Think about the definition of Herbrand Interpretation, where we say that I maps a constant to itself. What does it mean?

Exercise 5. Take the following formula F and put it in clause form.

$$\forall x(r(x) \to (\exists y \exists z(p(y) \wedge p(z) \wedge q(y,z,x))))$$

Let f and g the names of the newly introduced Skolem functions. Then, for each of the following interpretations of F, find the corresponding Herbrand interpretations.

$I_1 : D_1 = \mathbf{N}$
- $f(x) =$ the predecessor of x
- $g(x) =$ the integer division by 2 of x
- $p(x)$ means that x is prime
- $q(x,y,z)$ means that z is the sum of x and y
- $r(x)$ means that x is even and non-zero

$I_2 : D_2 = \{0,1,2,3,4,5\}$
- $f(x) =$ the successor of x
- $g(0) = 1$
- $g(x) = x$ multiplied by 5, modulo 6 (if $x \neq 0$)
- $p(x)$ means $x = 0$
- $q(x,y,z)$ means that z is x multiplied by y, *modulo 6*
- $r(x)$ means that $x \neq 0$

Consider the domain and the interpretation of predicates in I_1. Do you recognize the meaning of F under I_1 (regardless of how f and g are interpreted)? Does I_1 satisfy F? What is the meaning of f and g in the clause form, when they were not in F?

Exercise 6. Prove the following lemma: if an interpretation $\mathcal{I} = (D, I)$ satisfies F , then all Herbrand interpretations of F which correspond to $\mathcal{I}$ also satisfy F.

Hint: first try the case where F has constants, then when F does not.

Exercise 7. Find an example of a formula F and an interpretation I where I does not satisfy F but some corresponding I_H does.

Note: it is possible to find one even when F has constants.

Exercise 8. Find a rule for computing how many Herbrand interpretations correspond to a given (D, I), in the case where no constants appear in $\mathcal{C}$.

Exercise 9. Consider the clause $C = p(x) \vee q(x, f(x))$ and the interpretation

$$
\begin{aligned}
I_H = \{\; & \neg p(a), \neg p(f(a)), \neg p(f(f(a))), \ldots \\
& \neg q(a, a), q(a, f(a)), \neg q(a, f(f(a))), \ldots \\
& \neg q(f(a), a), q(f(a), f(a)), \neg q(f(a), f(f(a))), \ldots \\
& \ldots \}
\end{aligned}
$$

(1) Does I_H satisfy C? (2) Try to find an interpretation I on $\mathbf{N}$ (natural numbers) such that I_H possibly corresponds to I and I satisfies C iff I_H does.

Note: since the enumeration of I is not complete, there could be many I'. Just find a reasonable one.

Note: in order to do this exercise you somehow have to interpret the dots. Just do it freely but consistently!

Exercise 10. Consider the clauses $\mathcal{C} = \{p(x),\ q(f(f(y)))\}$ and the interpretation

$$
\begin{aligned}
I_H = \{\; & p(a), p(f(a)), p(f(f(a))), \ldots \\
& q(a), \neg q(f(a)), q(f(f(a))), q(f(f(f(a)))) \ldots \}
\end{aligned}
$$

(1) Does I_H satisfy C? (2) Try to find an interpretation I on $\mathbf{N}$ (natural numbers) such that I_H possibly corresponds to I and I satisfies C iff I_H does.

Note: since the enumeration of I is not complete, there could be many I'. Just find a reasonable one.

Note: in order to do this exercise you somehow have to interpret the dots. Just do it freely but consistently!