Computational Logic

Standardization of Formulæ

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Skolem normal form

Our goal: transforming formulæ

We want to obtain, by means of a series of *transformations*, a formula which is easier to deal with automatically, yet retains certain properties of the original one

- this is called *standardization*

Running example

$$\forall y \quad (\quad \exists x\, (\; p(x, f(y)) \rightarrow (q(x) \wedge q(z))\;) \quad \vee \quad \neg \forall w r(g(w), y) \quad)$$

Skolem normal form

How to get the Skolem Normal Form (SNF)

1. all quantifiers to the head of the formula (*prenex form*)
   - move quantifiers by means of equivalence rules
2. no free occurrences of variables
   - do the *existential closure*
3. the *matrix* of the formula is in *conjunctive normal form* (CNF): a conjunction of disjunctions of literals
   - transform the formula by means of equivalence rules
4. only universal quantifiers
   - remove existential quantifiers by introducing *Skolem functions*

What does this transformation preserve?

- it preserves the satisfiability
- but not all the models: the result is *not* equivalent to the original

Skolem normal form

1 Prenex form: all quantifiers at the beginning

Getting a prenex form relies on the following rules for *moving* quantifiers:

- renaming of bounded occurrences (if y does not occur in F)
  $\vdash \forall x F(x) \leftrightarrow \forall y F(x/y)$ $\quad\quad$ $\vdash \exists x F(x) \leftrightarrow \exists y F(x/y)$
- interdefintion of quantifiers
  $\vdash \neg\forall x F(x) \leftrightarrow \exists x \neg F(x)$ $\quad\quad$ $\vdash \neg\exists x F(x) \leftrightarrow \forall x \neg F(x)$
- connectives vs. quantifiers (if x does not appear in the other formula)
  $\vdash \forall x F \wedge G \leftrightarrow \forall x(F \wedge G)$ $\quad\quad$ $\vdash (\forall x F \rightarrow G) \leftrightarrow \exists x(F \rightarrow G)$
  $\vdash \exists x F \wedge G \leftrightarrow \exists x(F \wedge G)$ $\quad\quad$ $\vdash (\exists x F \rightarrow G) \leftrightarrow \forall x(F \rightarrow G)$
  $\vdash \forall x F \vee G \leftrightarrow \forall x(F \vee G)$ $\quad\quad$ $\vdash (F \rightarrow \forall x G) \leftrightarrow \forall x(F \rightarrow G)$
  $\vdash \exists x F \vee G \leftrightarrow \exists x(F \vee G)$ $\quad\quad$ $\vdash (F \rightarrow \exists x G) \leftrightarrow \exists x(F \rightarrow G)$
- connectives vs. quantifiers (more)
  $\vdash (\forall x F \wedge \forall x G) \leftrightarrow \forall x(F \wedge G)$ $\quad\quad$ $\vdash (\exists x F \vee \exists x G) \leftrightarrow \exists x(F \vee G)$

Skolem normal form

Lemma

The prenex form of a formula always exists, although it could be non-unique

Proof.

How would we prove it?

Lemma

Every formula F is equivalent to its prenex form(s):

$$\vdash F \leftrightarrow Prenex(F)$$

Proof.

Easy because all steps leading to $Prenex(F)$ are equivalencies

Skolem normal form

2 Existential closure: no more free variable occurrences

Variables which occur free in the formula are existentially quantified

$$\forall y(x \wedge q(y)) \rightsquigarrow \exists x(\forall y(x \wedge q(y)))$$
$$\forall y \exists x(p(x) \wedge q(y) \rightarrow r(f(z), x)) \rightsquigarrow \exists z \forall y \exists x(p(x) \wedge q(y) \rightarrow r(f(z), x))$$

Lemma

- *the closure does not affect satisfiability:* $F(x)$ *is satisfiable iff* $\exists x F(x)$ *is*
- *by extension,* $SAT(F)$ *iff* $SAT(\exists\text{–}closure(F))$

Skolem normal form

3. Conjunctive normal form (CNF): the *matrix* becomes a conjunction of disjunctions of literals

- connectives
  $\vdash (F \rightarrow G) \leftrightarrow (\neg F \vee G)$
  $\vdash (F \leftrightarrow G) \leftrightarrow (F \rightarrow G) \wedge (G \rightarrow F)$
- De Morgan
  $\vdash \neg(F \wedge G) \leftrightarrow \neg F \vee \neg G$ $\quad$ $\vdash \neg(F \vee G) \leftrightarrow \neg F \wedge \neg G$
- distributivity of $\wedge$ and $\vee$
  $\vdash F \wedge (G \vee H) \leftrightarrow (F \wedge G) \vee (F \wedge H)$
  $\vdash F \vee (G \wedge H) \leftrightarrow (F \vee G) \wedge (F \vee H)$

Skolem normal form

Lemma

The conjunctive normal form of a (quantifier-free) formula always exists

Proof.

(Exercise)

Lemma

For every formula F, $\vdash F \leftrightarrow CNF(F)$

Proof.

Easy because all steps leading to $CNF(F)$ are equivalencies

Skolem normal form

4. **∃-elimination:** no more existential quantifiers

An existential quantifier can be removed by replacing the variable it bounds by a *Skolem function* of the form $f(x_1,..x_n)$, where:

- f is a *fresh* function symbol
- $x_1,..,x_n$ are the variables which are universally quantified *before* the quantifier to be removed

$$
\begin{array}{lcl}
\forall x \exists y (p(x) \rightarrow \neg q(y)) & \rightsquigarrow & \forall x (p(x) \rightarrow \neg q(f(x))) \\
\exists x \forall z (q(x,z) \vee r(a,x)) & \rightsquigarrow & \forall z (q(b,z) \vee r(a,b)) \\
\exists x \forall y \exists z (p(x) \wedge q(y) \rightarrow r(f(z),x)) & \rightsquigarrow & \forall y (p(a) \wedge q(y) \rightarrow r(f(g(y)),a))
\end{array}
$$

Lemma

A formula F is satisfiable iff $Skolem(F)$ is

Skolem normal form

Definition

$$Q.M = \exists{-}closure(Prenex(F))$$
$$SNF(F) = Skolem(Q.CNF(M))$$

$Q.M = $ [quantifiers].[matrix]

Theorem

F is satisfiable iff $SNF(F)$ is

Proof.

1. F is satisfiable iff $Prenex(F)$ is
2. $Prenex(F)$ is satisfiable iff $\exists{-}closure(Prenex(F))$ is
3. M is satisfiable iff $CNF(M)$ is
4. $Q.M$ is satisfiable iff $Q.CNF(M)$ is (from ❸)
5. $Q.CNF(M)$ is satisfiable iff $Skolem(Q.CNF(M))$ is

Skolem normal form

Conclusion

- we are basically interested in satisfiability
- $SNF(F)$ exists for every F
- $SNF(F)$ preserves satisfiability

therefore, we can restrict ourselves to only formulæ in Skolem normal form

- the Skolem normal form is named after the Norwegian mathematician Thoralf Skolem
- it was introduced in this context by Davis and Putnam in 1960

Advantages

- no internal quantifiers
- only universal quantifiers, only in the head
- no free variable occurrences
- only $\wedge$ and $\vee$, suitably arranged

Clause form

It is easier to work on the *Clause Form* $CF(F)$

- *clause*: disjunction of literals
- the clause form of F is the set of clauses of $SNF(F)$, where the set means conjunction, and all variables are *universally quantified*

$$\begin{aligned} F &= \forall x(p(x) \wedge \forall y(\neg q(y) \rightarrow r(z,x))) \\ SNF(F) &= \forall x \forall y(p(x) \wedge (q(y) \vee r(a,x))) \\ CF(F) &= \{p(x), q(y) \vee r(a,x)\} \end{aligned}$$

Theorem

F is satisfiable iff $CF(F)$ is

Clause form

Clause form of a deduction

A deduction $[F_1, .., F_n] \vdash G$ is correct iff $F_1 \wedge .. \wedge F_n \wedge \neg G$ is not satisfiable

- get the clause form of every F_i
- get the clause form of $\neg G$
- compute the union of all sets of clauses
- check the satisfiability

Clause form

Example: $[\exists x f(x), \exists x g(x)] \vdash \exists x(f(x) \wedge g(x))$

$$\begin{array}{lcl} CF(\exists x f(x)) & = & f(a) \\ CF(\exists x g(x)) & = & g(b) \\ CF(\neg(\exists x(f(x) \wedge g(x)))) & = & \neg f(x) \vee \neg g(x) \end{array}$$

Here, there exists an interpretation which is a model:

- $D = \{0, 1\}$
- $I(a) = 0$
- $I(b) = 1$
- $I(f(a)) = I(g(b)) = \mathbf{t}$
- $I(f(b)) = I(g(a)) = \mathbf{f}$

therefore, the deduction is not correct