Computational Logic

Off-Topic Notes

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Academic Year 2008/2009

Constants, Names, Equality and Domains

Is Madonna a dancer?

When we say *b refers to Madonna* we mean:

- that b is a constant, i.e., a name
- that Madonna is not a name, but a real person! So that it does not make sense to ask *dancer*(Madonna)?: rather, we should ask $dancer(b)$?
- according to the usual meaning of equality, $a = b$ if both a and b refer to the same person: Madonna
- *Jim Henle is Madonna* means that they are the same person, which amounts to say $a = b$

On the other hand, the meaning of using 1 as a constant is that

- the constant 1 (i.e., the name 1) refers to the natural number 1
- $4 \neq 5$ because the name 4 denotes (refers to) the number 4, while the name 5 denotes the number 5

The second situation is equal to having a constant *Madonna* which denotes the person Madonna

Completeness and Gödel's first incompleteness theorem

Completeness vs. incompleteness

- the *completeness theorem* says that every logically valid formula is provable
- *Gödel's first incompleteness theorem* says that if an *effective* theory is *consistent* and expressive enough to describe *arithmetic*, then there is a formula F which is true but not provable in the theory
- both theorems can hold for the same theory

mmh... what are we missing here?

$(\neg$completeness$)\ \neq$ incompleteness

- the *completeness theorem* talks about fomulæ which are *logical consequences* of a theory
- the *Gödel's first incompleteness theorem* talks about a theory which cannot prove some F (which is *not* a logical consequence of the theory)

Completeness and Gödel's first incompleteness theorem

Peano's Arithmetic (first-order version)

1. $\forall n(nat(n) \rightarrow n = n)$
2. $\forall n \forall m((nat(n) \wedge nat(m) \wedge n = m) \rightarrow m = n)$
3. $\forall n \forall n' \forall n''((nat(n) \wedge nat(n') \wedge nat(n'') \wedge n = n' \wedge n' = n'') \rightarrow n = n'')$
4. $\forall a \forall b((nat(a) \wedge a = b) \rightarrow nat(b)))$
5. $nat(0)$
6. $\forall n(nat(n) \rightarrow nat(s(n)))$
7. $\forall n(nat(n) \rightarrow \neg(s(n) = 0))$
8. $\forall n \forall m((nat(n) \wedge nat(m) \wedge s(n) = s(m)) \rightarrow n = m)$
9. $\forall \bar{y}((\phi(0, \bar{y}) \wedge \forall n(\phi(n, \bar{y}) \rightarrow \phi(s(n), \bar{y}))) \rightarrow \forall m(\phi(m, \bar{y})))$

- every logical consequence of this theory is provable (see also Deduction theorem)
- there is a formula which is true but cannot be proven in the theory

Wrong deductions: Euclid's Fifth Postulate

A (2000+)-year-old problem (from 300 b.C. to XIX Century)

Given

1. a straight line segment can be drawn joining any two points
2. any straight line segment can be extended indefinitely in a straight line
3. given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center
4. all right angles are congruent

is it possible to prove

5. if two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough

?

Wrong deductions: Euclid's Fifth Postulate

Non-Euclidean geometries

- the mathematicians *Karl Friedrich Gauss*, *János Bolyai* and *Nikolai Ivanovich Lobachevsky* (Лобаче́вский) independently came to the conclusion that no proof exists
- there exist *models* of the first four postulates where the fifth postulate does not hold
  - *spherical geometry*
  - *hyperbolic geometry*
- is the fifth postulate true in the *real* world?
- in other words, is our geometry *euclidean* or *curved*?
  - logicians basically don't care, but philosophers do!
  - Einstein's theory of *general relativity* seems to give an answer...