Computational Logic

Standardization of Interpretations

Damiano Zanardini

UPM European Master in Computational Logic (EMCL)
School of Computer Science
Technical University of Madrid
damiano@fi.upm.es

Academic Year 2008/2009

Satisfiability and Interpretations

The problem

- F is unsatisfiable iff there is no interpretation $\mathcal{I}$ such that $\mathcal{I}(F) = \mathbf{t}$
- in order to check this, we should consider *all models*:
  - if F is propositional with n different propositions, then there are 2^n models
  - in a first order formula, the number of interpretations can be non-countable!
- it would be useful to have a subset of interpretations of F such that
  - it contains a smaller (finite or countable) number of interpretations
  - analyzing it is sufficient to decide the satisfiability of F
- such interpretations exist for every formula, and are called *Herbrand interpretations*

Satisfiability and Interpretations

Jacques Herbrand

- (Paris, France, February 12, 1908 - La Bérarde, Isère, France, July 27, 1931)
- PhD at École Normale Superieure, Paris, in 1929
- joined the army in October 1929
- *H. universe*, *H. base*, *H. interpretation*, *H. structure*, *H. quotient*
- *Herbrand's Theorem*: actually, two different results have this name
- introduced the notion of *recursive function*
- worked with John von Neumann and Emmy Noether
- died falling from a mountain in the Alps while climbing

Satisfiability and Interpretations

Jacques Herbrand

- (Paris, France, February 12, 1908 - La Bérarde, Isère, France, July 27, 1931)
- PhD at École Normale Superieure, Paris, in 1929
- joined the army in October 1929
- *H. universe*, *H. base*, *H. interpretation*, *H. structure*, *H. quotient*
- *Herbrand's Theorem*: actually, two different results have this name
- introduced the notion of *recursive function*
- worked with John von Neumann and Emmy Noether
- died falling from a mountain in the Alps while climbing

not exactly like this...

Herbrand Universe

Herbrand universe $H(F)$ of a formula F

- determines the domain of interpretation of F for Herbrand interpretations
- consists of all terms which can be formed with the constants and functions occurring in F

Herbrand universe: definition

$$
\begin{aligned}
Const(F) &= \text{set of constant symbols in } F \\
Fun(F) &= \text{set of function symbols in } F \\
H_0 &= \begin{cases} Const(F) & Const(F) \neq \emptyset \\ \{a\} & Const(F) = \emptyset \end{cases} \\
H_{i+1} &= \{f(t_1, .., t_n) \mid t_j \in (H_0 \cup .. \cup H_i),\ f/n \in Fun(F)\} \\
H(F) &= H_0 \cup .. \cup H_i \cup .. \qquad \text{is the Herbrand universe}
\end{aligned}
$$

Herbrand Universe

Herbrand universe: examples

- $F = \{p(x),\ q(y)\}$
  - $H_0 = \{a\}$
  - $H_1 = H_2 = .. = \emptyset$
  - $H(F) = \{a\}$
- $F = \{p(x, a),\ q(y) \vee \neg r(b, f(x))\}$
  - $H_0 = \{a, b\}$
  - $H_1 = \{f(a), f(b)\}$
  - $H_2 = \{f(f(a)), f(f(b))\}$
  - ...
  - $H(F) = \{a, b, f(a), f(b), f(f(a)), f(f(b)), f(f(f(a))), f(f(f(b))), ..\} = \{f^n(a), f^n(b)\}_{n \geq 0}$

Herbrand Base

Herbrand base of F

- *ground atom*: an atom which is obtained by applying a predicate symbol of F to a term t in the Herbrand universe of F
- the *Herbrand base* of F is the set of all the possible ground atoms of F

Herbrand base: definition

$Pred(F)$ is the set of predicate symbols in F

$$HB(F) = \{p(t_1, .., t_n) \mid t_j \in H(F)\,,\ p/n \in Pred(F)\}$$

Herbrand Base

Herbrand base: examples

- $F = \{p(x),\ q(y)\}$
  - $H(F) = \{a\}$
  - $HB(F) = \{p(a), q(a)\}$
- $F = \{p(a),\ q(y) \vee \neg p(f(x))\}$
  - $H(F) = \{a, f(a), f(f(a)), ..\} = \{f^n(a)\}_{n \geq 0}$
  - $HB(F) = \{p(a), p(f(a)), p(f(f(a))), .., q(a), q(f(a)), q(f(f(a))), ..\} = \cup(\{\{p(t), q(t)\} |\ t \in H(f)\})$
- $F = \{p(a),\ q(y) \vee \neg r(b, f(x))\}$
  - $H(F) = \{a, b, f(a), f(b), f(f(a)), f(f(b)), ..\} = \{f^n(a), f^n(b)\}_{n \geq 0}$
  - $HB(F) = \cup(\{\{p(t), q(t), r(t, t')\} |\ t, t' \in H(F)\})$

Herbrand Interpretations

A Herbrand interpretation of F

is an interpretation $\mathcal{I}_H = (H(F), I_H)$ on $H(F)$ such that:

- every constant $a \in Const(F)$ is assigned to itself: $I_H(a) = a$
- every function symbol $f/n \in Fun(F)$ is assigned to $I_H(f/n) = \mathcal{F} : (H(F))^n \mapsto H(F)$, such that
  - $\mathcal{F}(u_1, .., u_n) = f(u_1, .., u_n) \in H(F)$ where $u_i \in H(F)$
- every predicate symbol $p/n \in Pred(F)$ is assigned as $I_H(p/n) = \mathcal{P} : (H(F))^n \mapsto \{\mathbf{t}, \mathbf{f}\}$, such that
  - $I_H(p(u_1, .., u_n)) = \mathcal{P}(I_H(u_1), .., I_H(u_n)) = \mathcal{P}(u_1, .., u_n) \in \{\mathbf{t}, \mathbf{f}\}$
- every (ground) atom of $HB(F)$ has a truth value. Which one? It is *not* required by the definition

Herbrand Interpretations

Herbrand interpretations: notation

A Herbrand interpretation can be represented as the set of ground atoms in $HB(F)$: positive if they are interpreted as true, negative otherwise

$$
\begin{array}{rcll}
HB(F) & = & \{A_1, A_2, A_3, ..\} & \\
\mathcal{I}_H & = & \{A_1, \neg A_2, \neg A_3, ..\} & \text{if} \quad I_H(A_1) = \mathbf{t}, \\
 & & & \phantom{\text{if} \quad} I_H(A_2) = \mathbf{f}, \\
 & & & \phantom{\text{if} \quad} I_H(A_3) = \mathbf{f}, ..
\end{array}
$$

Terminology

- the notions of Herbrand universe, base, and interpretations will often refer to a *set of clauses*, written as $\mathcal{C}$, which can be actually the result of the standardization of a generic formula F
- in practice, F will be usually taken to be in clause form
- because we (computational logicians) are smarter than formal logicians?

Herbrand Interpretations

Herbrand interpretations: examples

- $F = \{p(x),\ q(y)\}$
  - $H(F) = \{a\}, \qquad HB(F) = \{p(a), q(a)\}$
  - there are 4 possible Herbrand interpretations:

$$\begin{array}{rclcrcl} \mathcal{I}_H^1 & = & \{p(a), q(a)\} & \qquad & \mathcal{I}_H^2 & = & \{p(a), \neg q(a)\} \\ \mathcal{I}_H^3 & = & \{\neg p(a), q(a)\} & & \mathcal{I}_H^4 & = & \{\neg p(a), \neg q(a)\} \end{array}$$

- $F = \{p(a),\ q(y) \vee \neg p(f(x))\}$
  - $H(F) = \{f^n(a)\}_{n \geq 0}, \qquad HB(F) = \cup(\{\{p(t), q(t)\} |\ t \in H(F)\})$
  - there are an infinite (how many?) number of Herbrand interpretations

$$\begin{array}{rcl} \mathcal{I}_H^1 & = & \cup(\{\{p(t), q(t)\} \mid t \in H(F)\}) \\ \mathcal{I}_H^2 & = & \{p(a)\} \cup \{\neg p(t) \mid t \in H(F) \setminus \{a\}\} \cup \{q(t) \mid t \in H(F)\} \\ \mathcal{I}_H^3 & = & \{p(t) \mid t \in H(F)\} \cup \{\neg q(t) \mid t \in H(F)\} \quad \ldots \end{array}$$

Herbrand Interpretations

Ground instances

A *ground instance* of a clause is a formula, in clause form, which results from replacing the variables of the clause by terms from its Herbrand universe

- by means of a Herbrand interpretation, it is possible to give a truth value to a formula starting from the truth value of its ground instances

Example: $F = \{p(a),\ q(b) \vee \neg p(x)\}$

- $H(F) = \{a, b\}$ $\qquad HB(F) = \{p(a), p(b), q(a), q(b)\}$
- $\mathcal{I}_H = \{p(a), \neg p(b), q(a), \neg q(b)\}$
- the first clause is true since its only instance, $p(a)$, is true in $\mathcal{I}_H$
- the second clause is false since one instance, $q(b) \vee \neg p(b)$, is true in $\mathcal{I}_H$, but the other, $q(b) \vee \neg p(a)$, is false

since F is the conjunction of both clauses, it is false for $\mathcal{I}_H$

Herbrand Interpretations

$\mathcal{I}_H$ corresponding to $\mathcal{I}$

Given $\mathcal{I} = (D, I)$, a Herbrand interpretation $\mathcal{I}_H = (D_H, I_H)$ *corresponds to* $\mathcal{I}$ for F if it satisfies the following condition:

- I' is a total mapping from $H(F)$ to D, such that
  - $I'(c) = d$ if $I(c) = d$ (constants)
  - $I'(f(t_1, .., t_n)) = \mathcal{F}(I'(t_1), .., I'(t_n))$ where $I(f/n) = \mathcal{F}/n$
- for every ground atom $p(t_1, .., t_n) \in HB(F)$, $I_H(p(t_1, .., t_n)) = \mathbf{t}$ (resp., $\mathbf{f}$) if $I(p)(I'(t_1), .., I'(t_n)) = \mathbf{t}$ (resp., $\mathbf{f}$)

This definition may look overly complicated, but simpler ones can be imprecise...

- let $h_1, .., h_n$ be elements of $H(F)$
- let every h_i be mapped to some $d_i \in D$
- if $p(d_1, .., d_n)$ is assigned $\mathbf{t}$ (resp., $\mathbf{f}$) by I, then $p(h_1, .., h_n)$ is also assigned $\mathbf{t}$ (resp., $\mathbf{f}$) by I_H
- [Chang and Lee. Symbolic Logic and Mechanical Theorem Proving]

Herbrand Interpretations

Example: $F = \{p(x), q(y, f(y, a))\}$, $D = \{1, 2\}$

- $I(a) = 2$
- $I(f/2) = \mathcal{F}/2$: $\mathcal{F}(1,1) = 1$ $\mathcal{F}(1,2) = 1$ $\mathcal{F}(2,1) = 2$ $\mathcal{F}(2,2) = 1$
- $I(p/1) = \mathcal{P}/1$: $\mathcal{P}(1) = \mathbf{t}$ $\mathcal{P}(2) = \mathbf{f}$
- $I(q/2) = \mathcal{Q}/2$: $\mathcal{Q}(1,1) = \mathbf{f}$ $\mathcal{Q}(1,2) = \mathbf{t}$ $\mathcal{Q}(2,1) = \mathbf{f}$ $\mathcal{Q}(2,2) = \mathbf{t}$

In this case, I' comes to be I

- $I_H(p(a)) = I(p(a)) = \mathcal{P}(I(a)) = \mathcal{P}(2) = \mathbf{f}$
- $I_H(q(a,a)) = I(q(a,a)) = \mathcal{Q}(I(a), I(a)) = \mathcal{Q}(2,2) = \mathbf{t}$
- $I_H(p(f(a,a))) = I(p(f(a,a))) = \mathcal{P}(\mathcal{F}(2,2)) = \mathcal{P}(1) = \mathbf{t}$
- ...

Herbrand Interpretations

Multiple Herbrand interpretations

There can be more than one corresponding $\mathcal{I}_H$ when F has no constants. In this case, there is no I-interpretation of H_0 (i.e., $I' \neq I$), so that the I_H-interpretation of $a \in H_0$ is arbitrary.:

- $F = \{p(x)\}, \quad D = \{1, 2\}, \quad p(x)$ means that x is even
- $H(F) = \{a\}, \quad HB(F) = \{p(a)\}$
- $I'(a) = 1$ and $I'(a) = 2$ are both legal
- $\mathcal{I}_H^1 = \{\neg p(a)\}$ supposing $a \rightsquigarrow 1$
- $\mathcal{I}_H^2 = \{p(a)\}$ supposing $a \rightsquigarrow 2$

Lemma

If an interpretation $\mathcal{I} = (D, I)$ satisfies F, then all Herbrand interpretations of F which correspond to $\mathcal{I}$ also satisfy F

- *Ex.* $F = \forall x p(x) \wedge \forall x q(f(x))$

Herbrand Interpretations

Theorem

A formula F is unsatisfiable iff it is false for all its Herbrand interpretations

Proof ($\rightarrow$).

1. F is unsatisfiable
2. it is false for every interpretation on every domain
3. in particular, all Herbrand interpretations make it false

Herbrand Interpretations

Theorem

A formula F is unsatisfiable iff it is false for all its Herbrand interpretations

Proof ($\longleftarrow$).

❶ F is false for all Herbrand interpretations

❷ suppose F be not unsatisfiable

❸ there exists an interpretation $\mathcal{I}$ satisfying F (from ❷)

❹ for the previous lemma, the corresponding Herbrand interpretations also satisfy F

❺ contradiction between ❶ and ❹, therefore ❷ is false

❻ F is unsatisfiable (from ❺)

Herbrand Interpretations

In practice

In order to study the unsatisfiability of a formula F, it is enough to study the Herbrand interpretations of its clause form $CF(F)$

For every Herbrand interpretation of $CF(F)$

- compute the ground instances of the clauses
- assign a truth value to every instance
- $CF(F)$ is true iff *every* ground instance of *every* clause is true

F is satisfiable iff *some* Herbrand interpretation makes $CF(F)$ true

Example: $F = \{p(x),\ q(y)\}$

- $H(F) = \{a\}$ $\qquad HB(F) = \{p(a), q(a)\}$

There are 4 Herbrand interpretations

- $\mathcal{I}_H^1 = \{p(a), q(a)\}$
- $\mathcal{I}_H^2 = \{p(a), \neg q(a)\}$
- $\mathcal{I}_H^3 = \{\neg p(a), q(a)\}$
- $\mathcal{I}_H^4 = \{\neg p(a), \neg q(a)\}$

Ground instances: $\{p(a), q(a)\}$

- $\mathcal{I}_H^1$ is a model since it verifies both instances
- $\mathcal{I}_H^2$, $\mathcal{I}_H^3$ and $\mathcal{I}_H^4$ are countermodels since they falsify at least one instance

Therefore, F is satisfiable

Herbrand Interpretations

Example: $F = \{p(y),\ q(a) \vee \neg p(f(x)),\ \neg q(x)\}$

- $H(F) = \{f^n(a) \mid n \geq 0\}$
  $HB(F) = \{p(t) \mid t \in H(F)\} \cup \{q(t) \mid t \in H(F)\}$

There are infinite Herbrand interpretations. For example

- $\mathcal{I}_H^1 = \{p(t) \mid t \in H(F)\} \cup \{q(t) \mid t \in H(F)\}$
- $\mathcal{I}_H^2 = \{q(a)\} \cup \{\neg q(t) \mid t \in H(F) \setminus \{a\}\} \cup \{p(t) \mid t \in H(F)\}$

Ground instances

$$\begin{array}{rcl} p(y) & \rightsquigarrow & p(a), p(f(a)), p(f(f(a))), .. \\ q(a) \vee \neg p(f(x)) & \rightsquigarrow & q(a) \vee \neg p(f(a)), q(a) \vee \neg p(f(f(a))), .. \\ \neg q(x) & \rightsquigarrow & \neg q(a), \neg q(f(a)), .. \end{array}$$

Every Herbrand interpretation falsify at least one instance, so that F is unsatisfiable