On the limits of the Classical Approach to Cost Analysis

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What is Cost Analysis?

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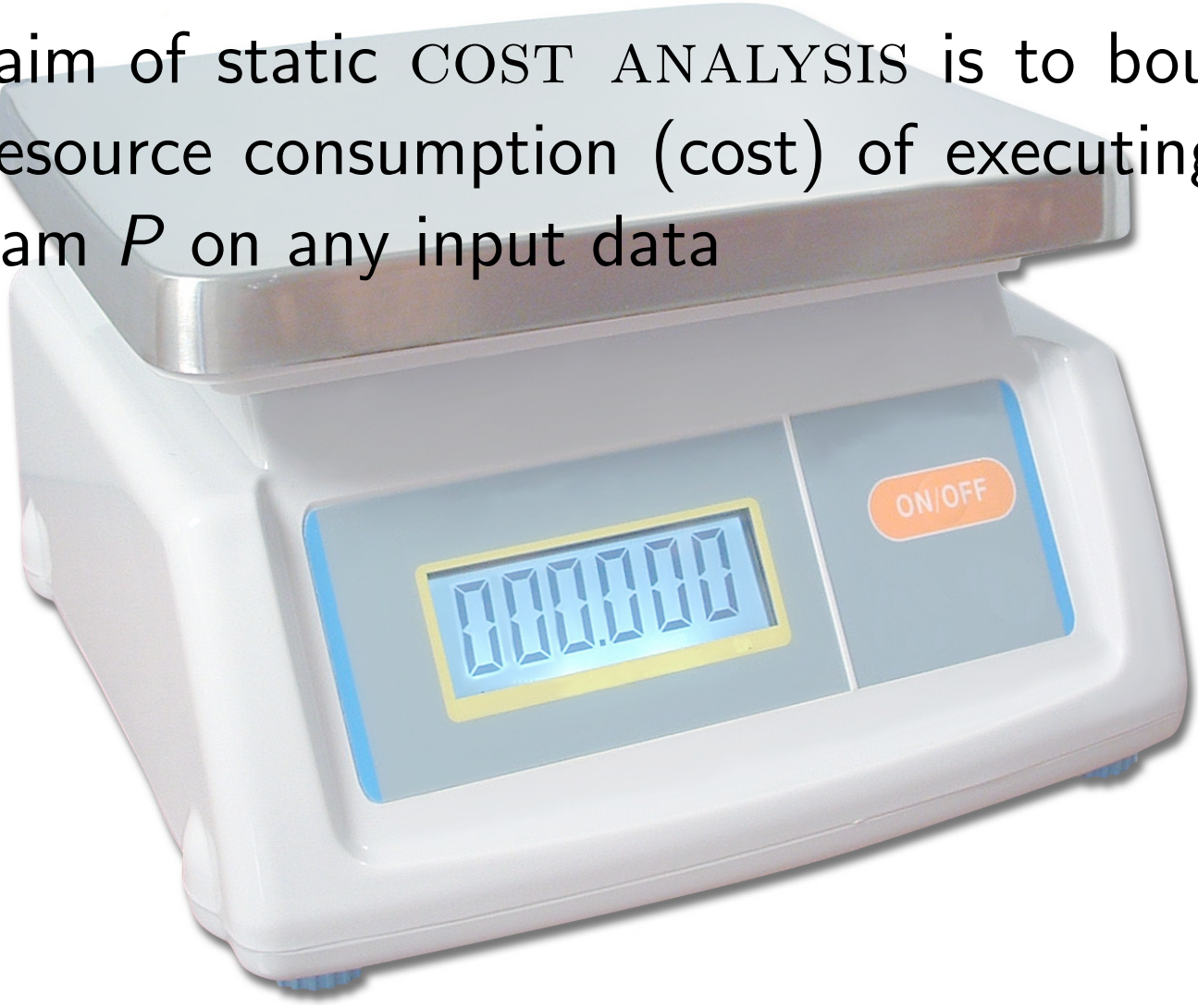
The aim of static COST ANALYSIS is to bound the resource consumption (cost) of executing a program P on any input data



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Bound functions

- Upper Bounds (*worst case*)
- Lower Bounds (*best case*)
- Non-Asymptotic: $P(x) = 2x^2 + 3$
- Asymptotic: $P^u(x) \in \mathcal{O}(x^2)$

What is Cost Analysis?

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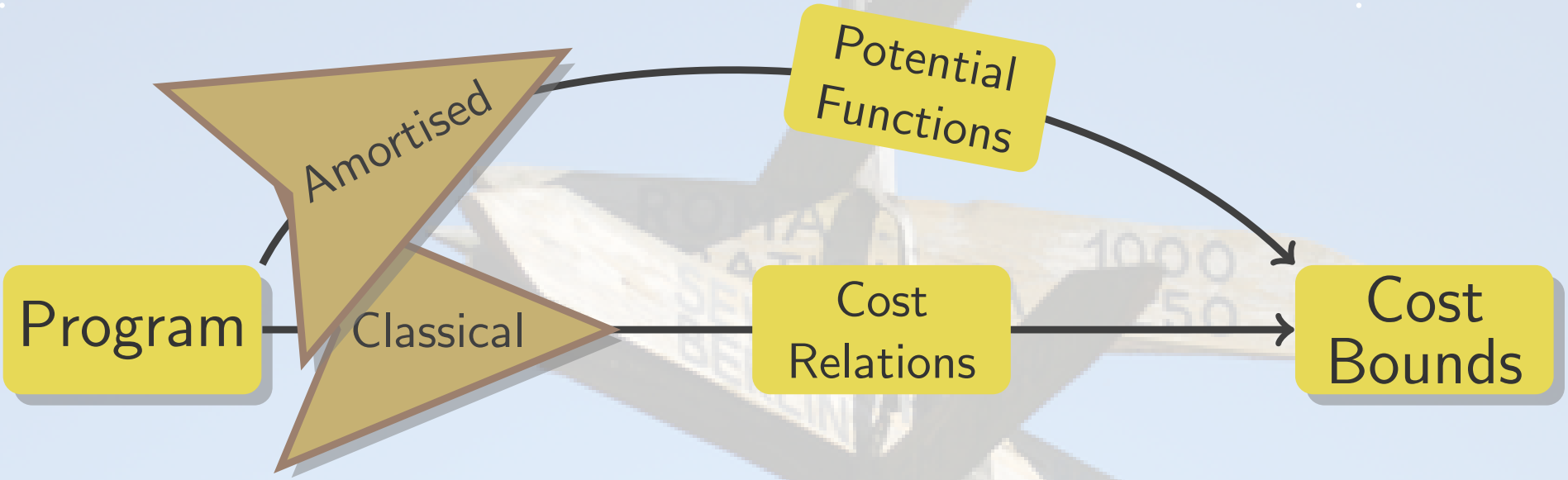

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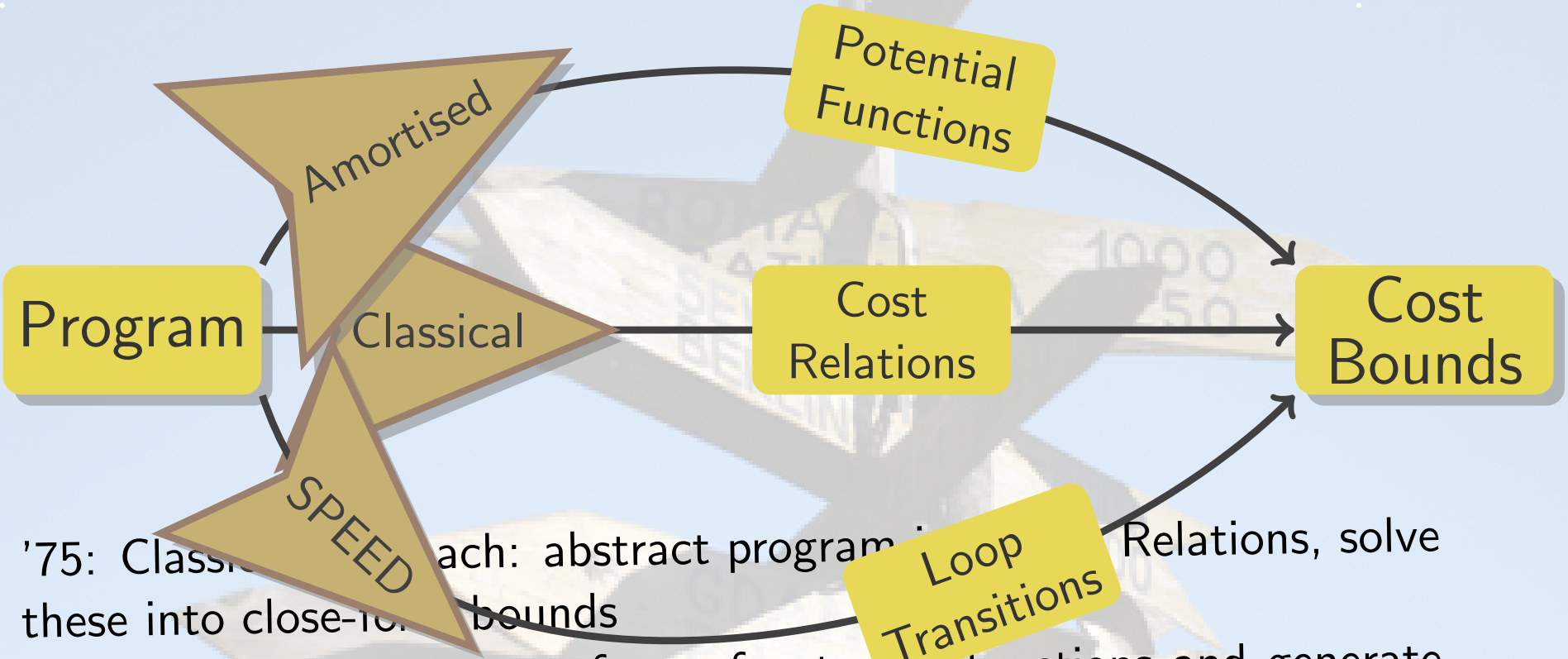

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'08: SPEED: abstract individual loop transitions, generate bounds for each transition, combine bounds

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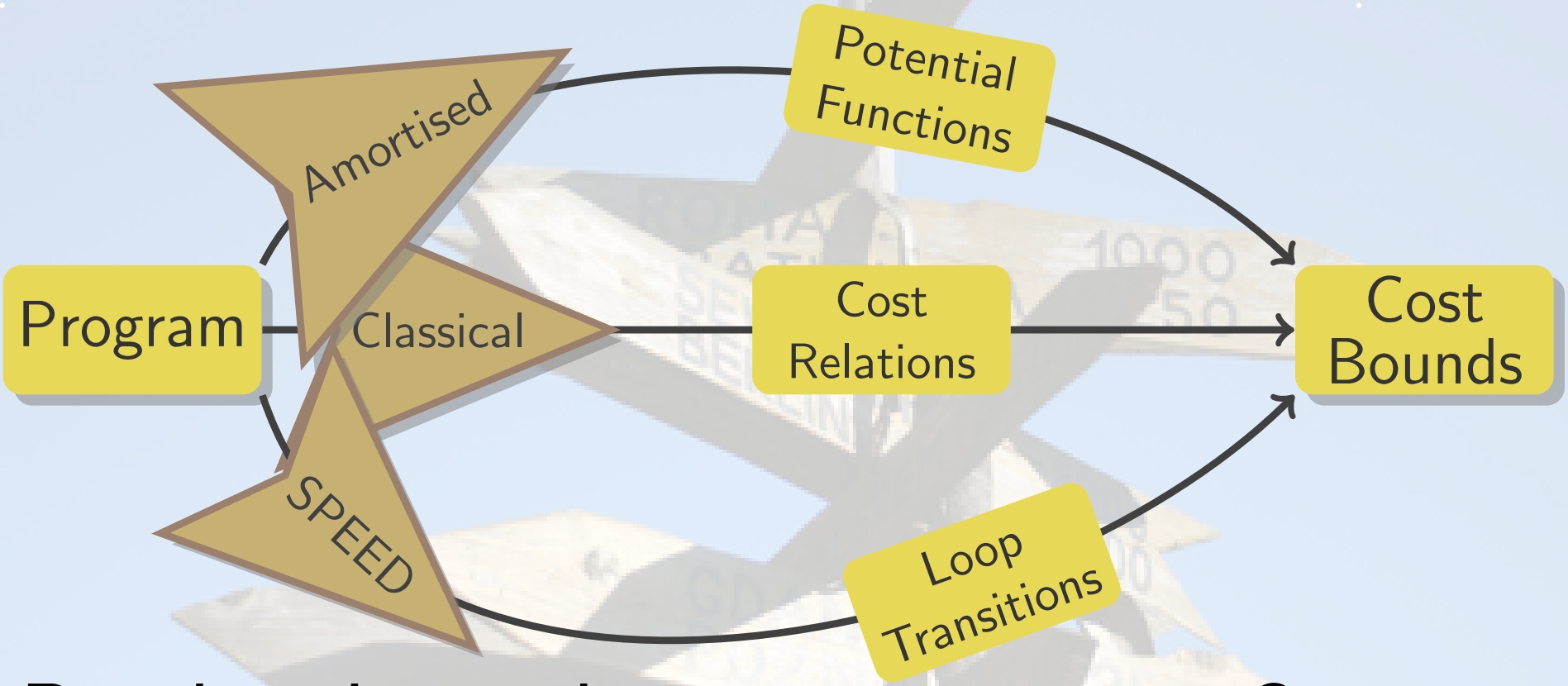

Do they have the same precission?

Can they handle same programs?

Cost Analysis: A working example

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class Stack {
  Node top;

  //@requires m >= 1
  void rpop(int m){
    if (top != null && *){
      top=top.next; //pop
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  }

  //@requires m >= 0
  void main(int m) {
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Upper Bounds (Worst Case)

$$rpop^u(t, m) = m$$
$$main^u(t, m) = \frac{m^2+m}{2} + m$$

Meaning of an Upper Bound

Executing `main[this.top, m]`, takes at most $m^2 + m$ resources

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Inferred upper bounds (UB)

$$rpop^u(t,m) = m * t$$
$$main^u(t,m) = \frac{m^3+3t(m^2+m)+5m}{6}$$

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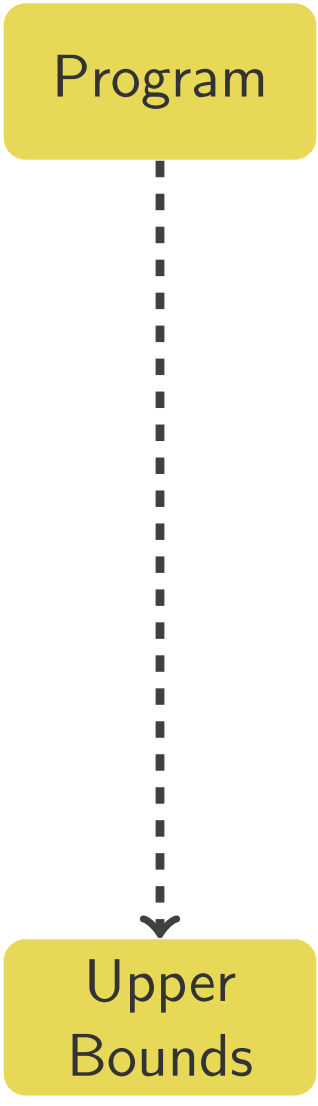

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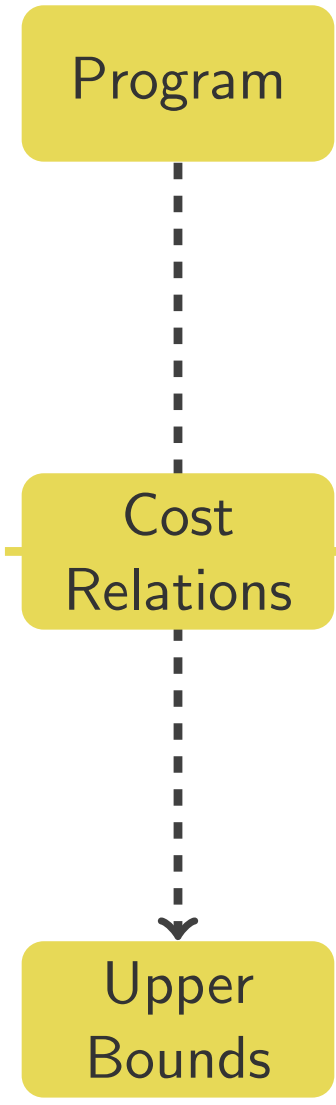

The Classical approach uses Cost Relations to split Cost Analysis in two independent phases

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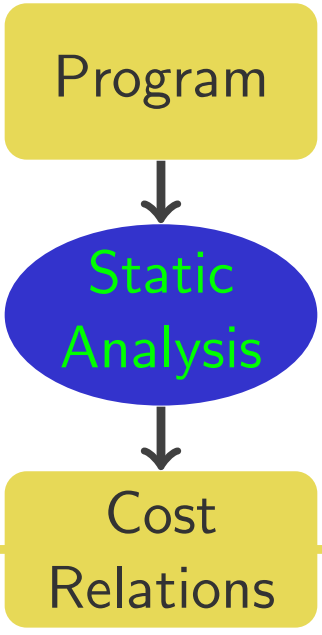

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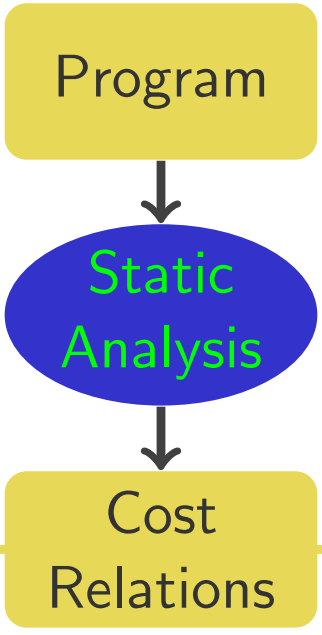

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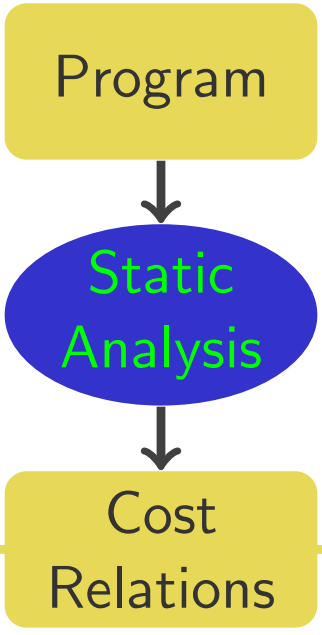

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Cost Relations have a recursive syntax, and a purely numerical yet undecidable semantics as $\mathbb{Z}^* \times \mathbb{R}^+$

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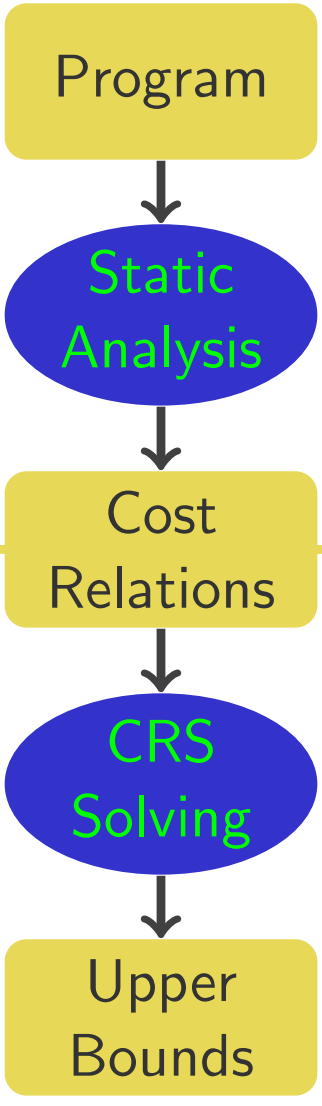

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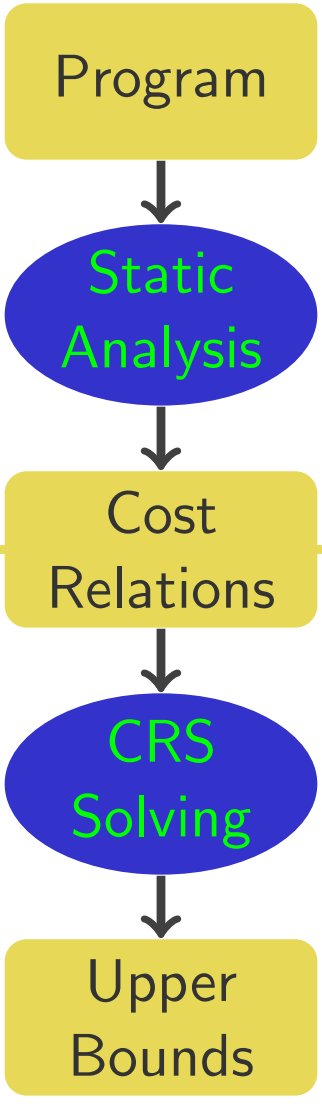

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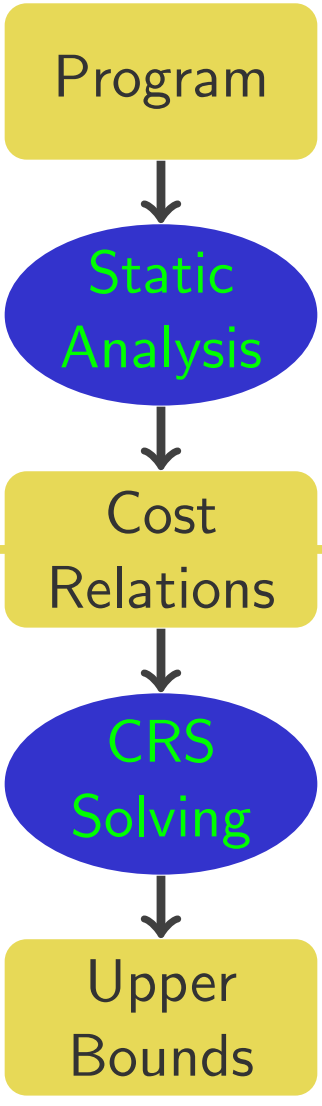

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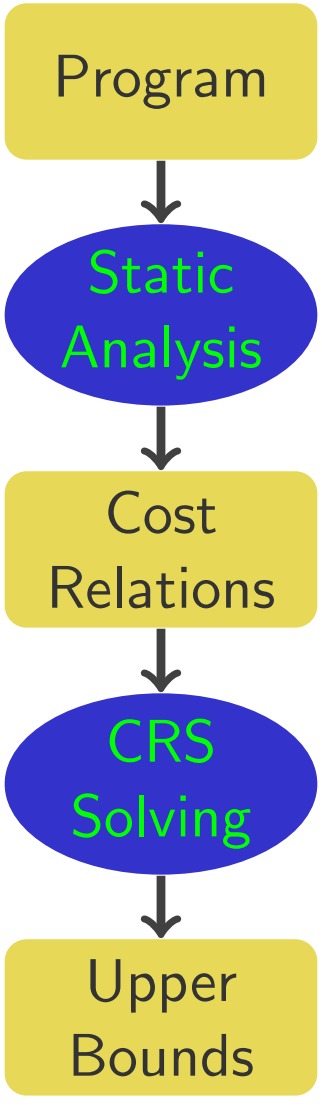

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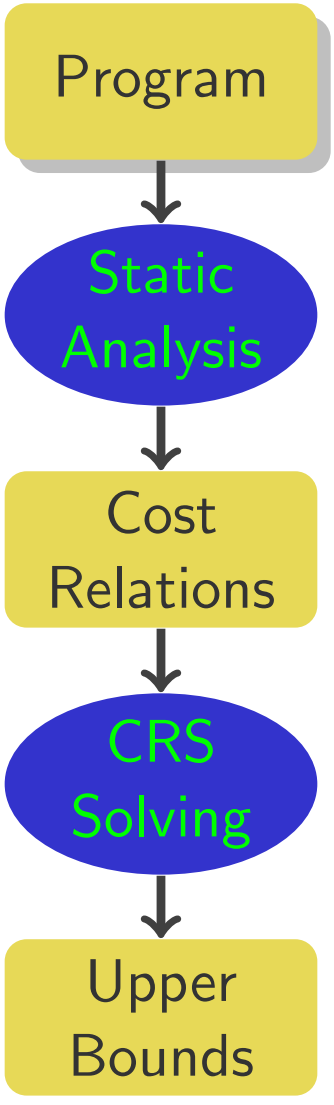


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//@requires m >= 0
void main(int m) {
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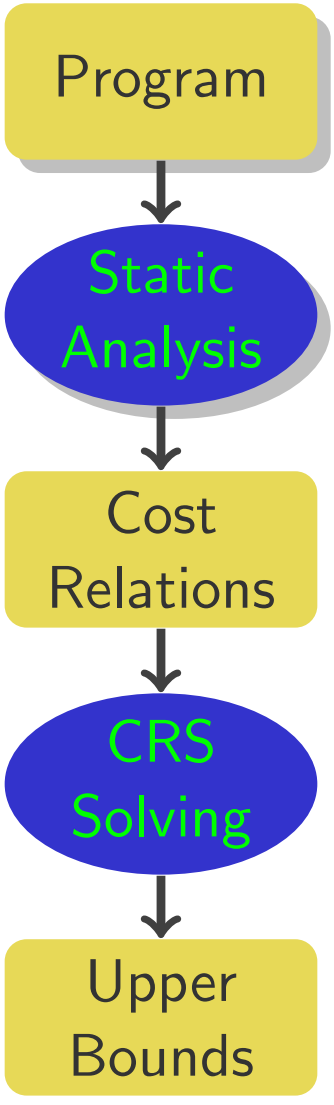


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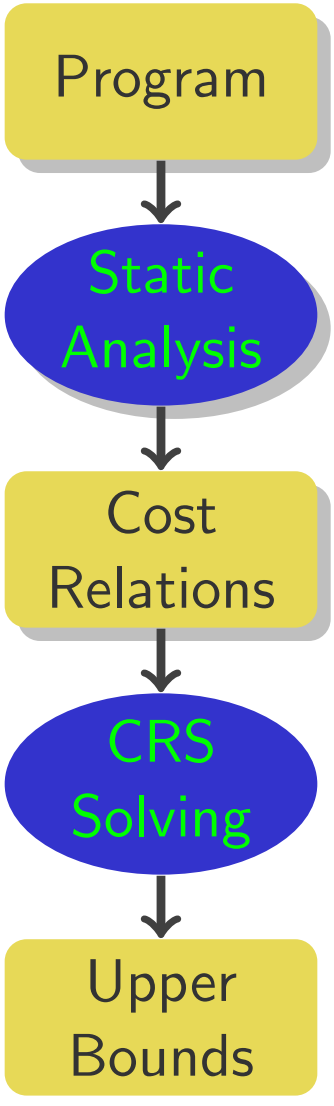


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$$main(t,m) = 0 \quad \{m = 0, t \geq 0\}$$

$$main(t,m) = \begin{matrix} rpop(t,m) + 1 + \\ main(t_2, m_1) \end{matrix} \quad \left\{ \begin{matrix} m \geq 1, t \geq 0, m_1 = m - 1 \\ t_2 = t' + 1, t \geq t' \geq t - 1 \end{matrix} \right\}$$

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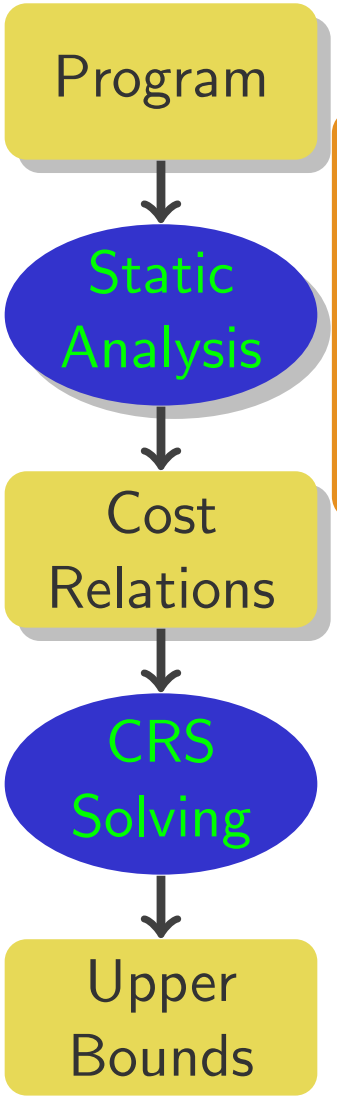

main : $\mathbb{Z}^2 \mapsto \mathcal{P}(\mathbb{R})$

$main(t, m)$ contains possible costs of executing `main(this.top,m)` if `(this.top,m)` have sizes (t, m)

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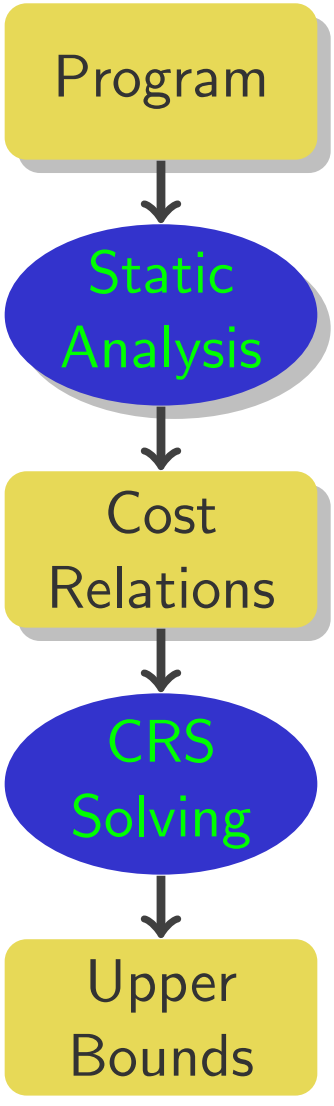


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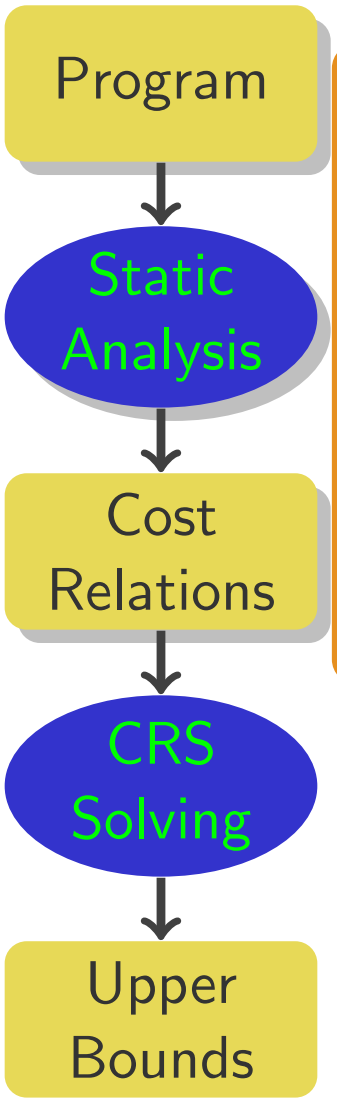

Modular definition

Cost Relations define the cost of executing a program fragment in terms of other fragments cost

- Control flow $\rightarrow$ **Calls**
- Guards $\rightarrow$ **Constraints**
- Iteration $\rightarrow$ **Recursion**

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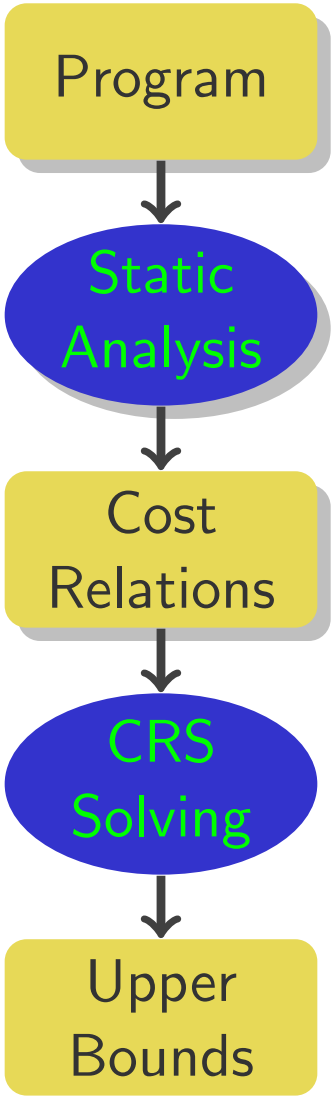


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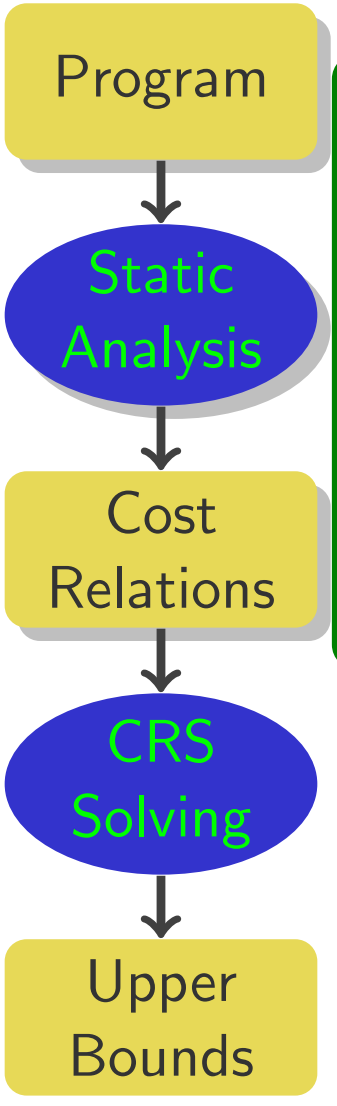

Size analysis

Global size analysis infers how data sizes change along execution.

Data	$\rightarrow$	**Sizes**
States	$\rightarrow$	**SSA form**
Instructions	$\rightarrow$	**Constraints**
Calls	$\rightarrow$	**Postconditions**

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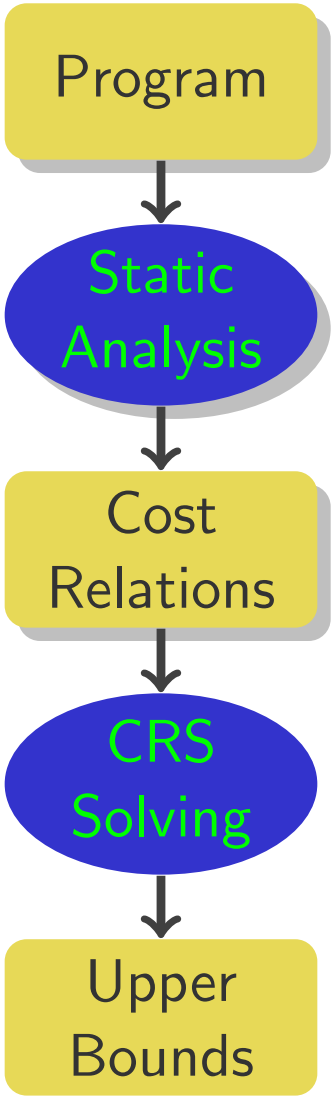
main(t,m) &= rpop(t,m) + 1 + main(t_2, m_1) && \left\{\begin{array}{l} m \geq 1, t \geq 0, m_1 = m - 1 \\ t_2 = t' + 1, t \geq t' > t - 1 \end{array}\right\}\\
rpop(t,m) &= 0 && \{m \geq 1\}\\
rpop(t,m) &= m && \{m \geq 1, t \geq 1\}
\end{aligned}
$$

Classical Approach to Cost Analysis: working example

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//@requires m >= 1
void rpop(int m){
  if (top != null && *){
    //pop
    top=top.next ;
    //@acquire(m)
}}
```

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//@requires m >= 0
void main(int m) {
  while( m > 0 ) {
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    // push(*);
    top = new
     Node(*,top);
    //@acquire(1)
    m = m-1      ;
}}}
```

$$main(t,m) = 0 \quad \{m = 0, t \geq 0\}$$

$$main(t,m) = \begin{matrix} rpop(t,m) + 1 + \\ main(t_2, m_1) \end{matrix} \quad \left\{ \begin{matrix} m \geq 1, t \geq 0, m_1 = m - 1 \\ t_2 = t' + 1, t \geq t' \geq t - 1 \end{matrix} \right\}$$

$$rpop(t,m) = 0 \quad \{m \geq 1\}$$

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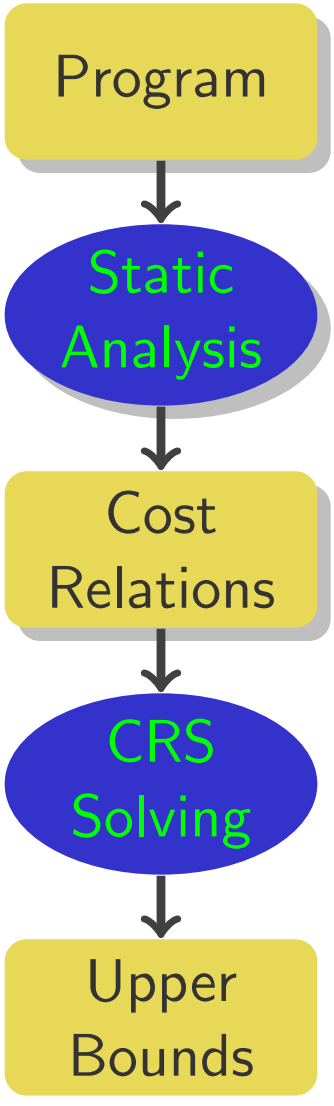


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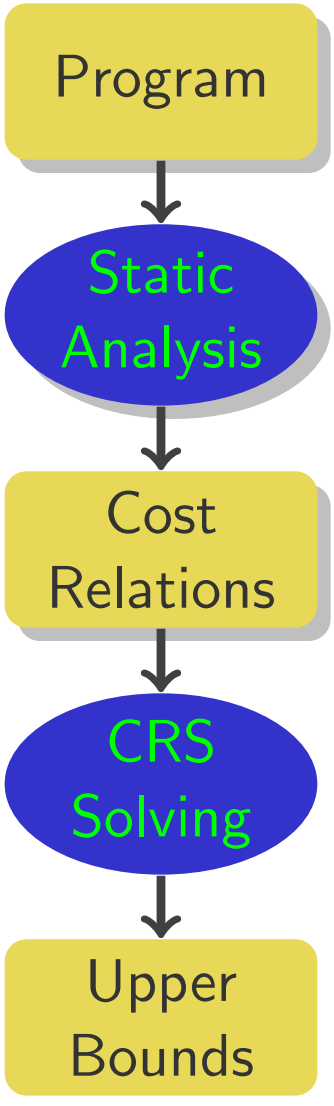


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//@requires m >= 1
void rpop(int m){

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void main(int m) {
```

Soundness & Precission of Analysis

Sound: For every program execution there is a CR evaluation with same cost

Precise: Every CR evaluation matches at least one program execution

$$main(t,m) = 0 \quad \{m = 0, t \geq 0\}$$

$$main(t,m) = \begin{matrix} rpop(t,m) + 1 + \\ main(t_2, m_1) \end{matrix} \quad \left\{ \begin{matrix} m \geq 1, t \geq 0, m_1 = m - 1 \\ t_2 = t' + 1, t \geq t' \geq t - 1 \end{matrix} \right\}$$

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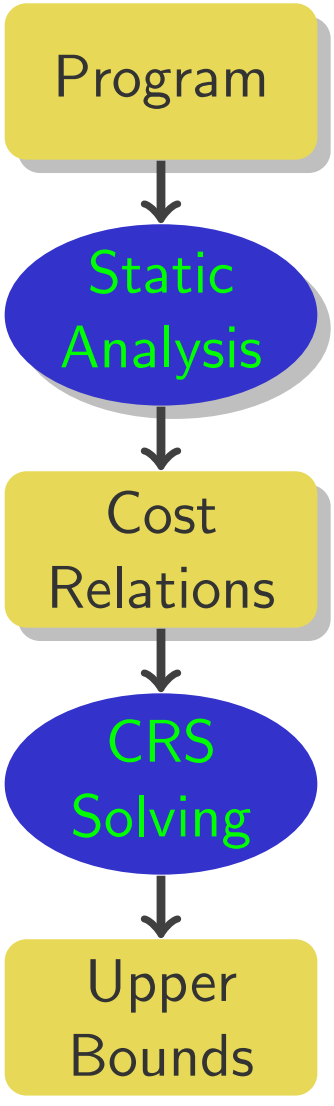


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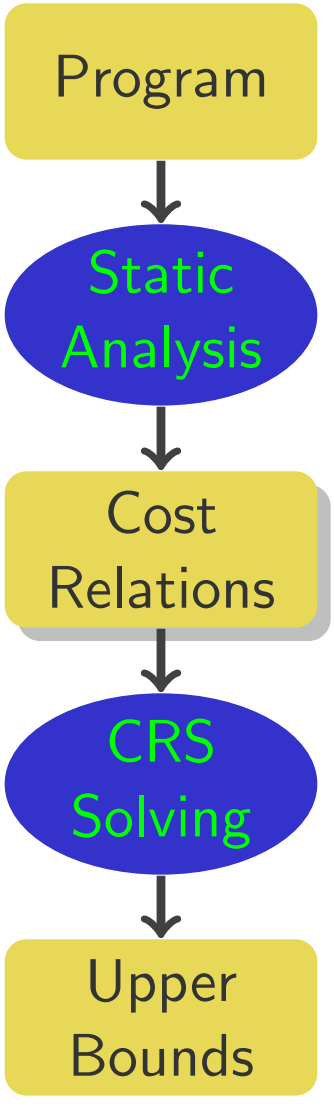


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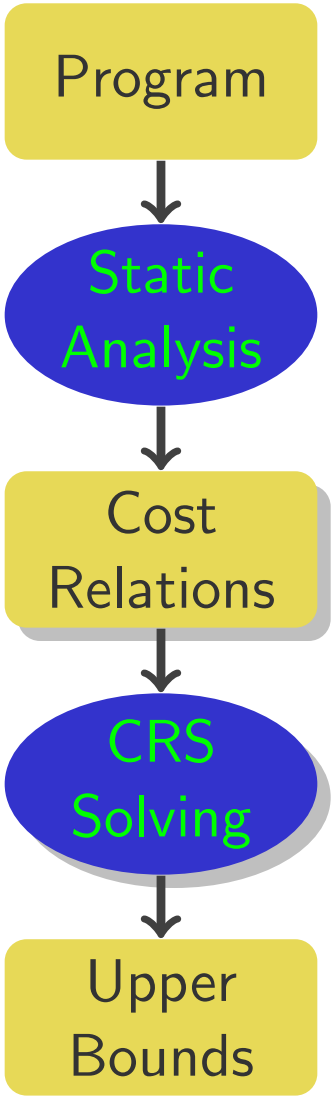


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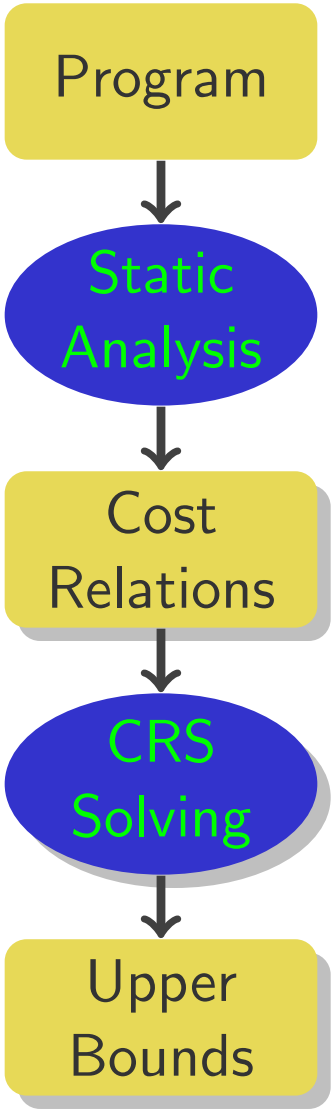


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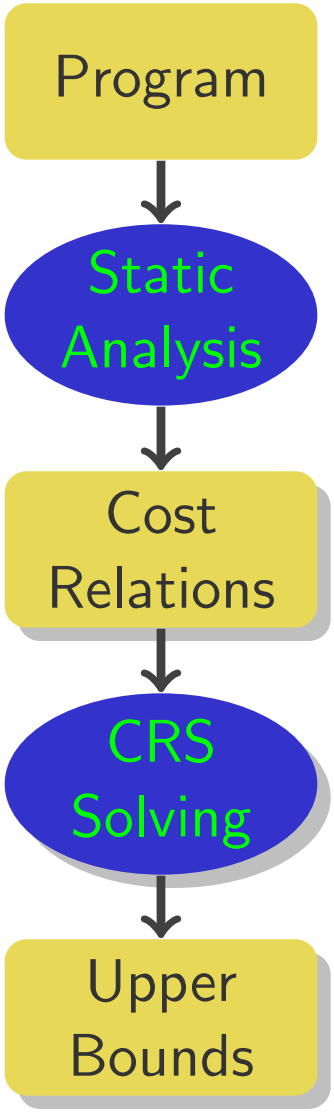
Classical Approach to Cost Analysis: working example




```
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void main(int m) {
```

Soundness & Precission of UBs wrt CRs

Sound: **If** $c \in main(t, m)$ **then** $c \leq main^u(t, m)$
Precise: **If** $main^u(t, m) = c$ **then** $c \in main(t, m)$

$$main^u(t, m) = \frac{m^2 + m}{2} + m \quad rpop^u(t, m) = m$$

$$main(t, m) = 0 \quad \{m = 0, t \geq 0\}$$

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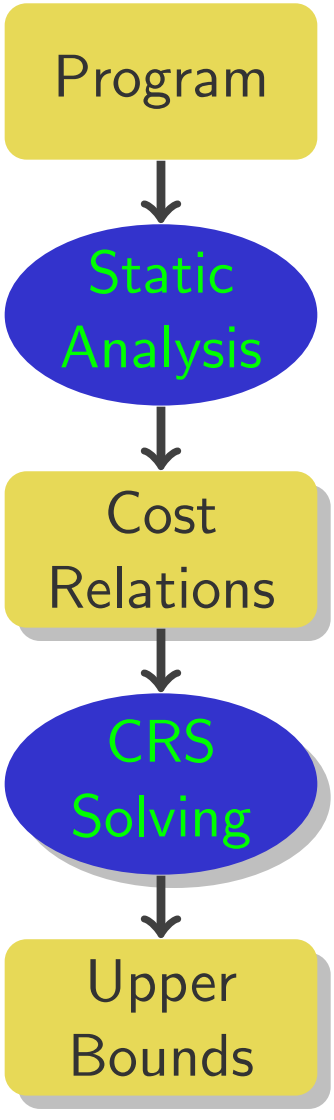
Classical Approach to Cost Analysis: working example

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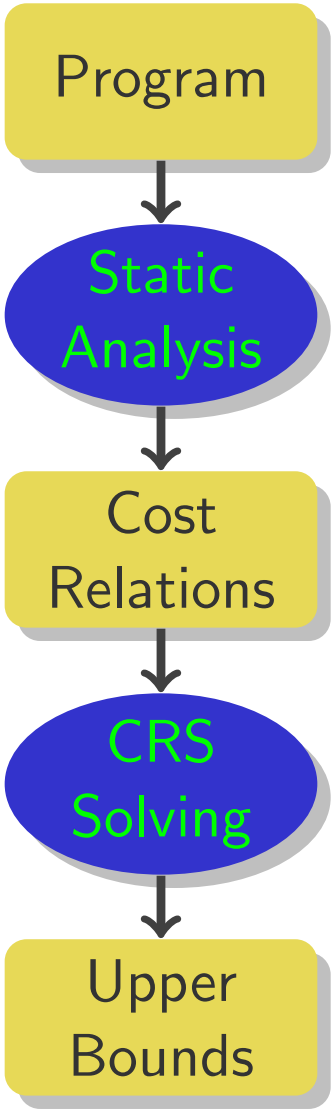


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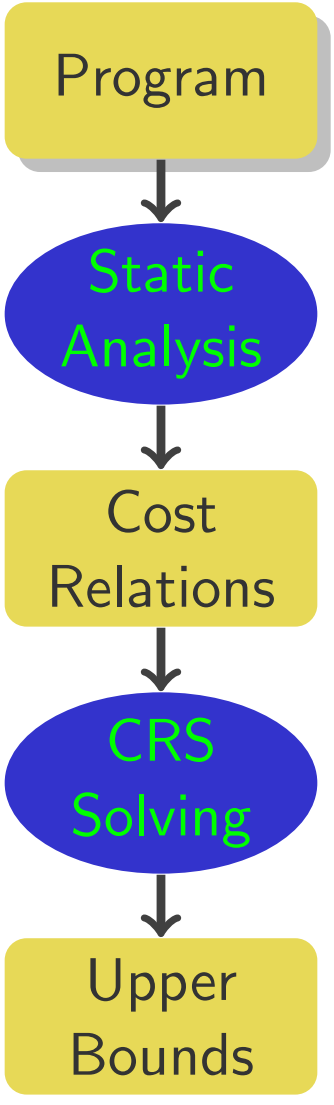


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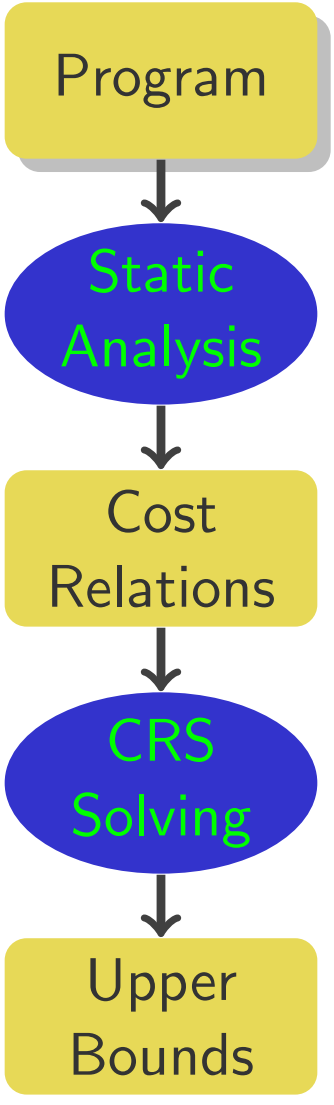


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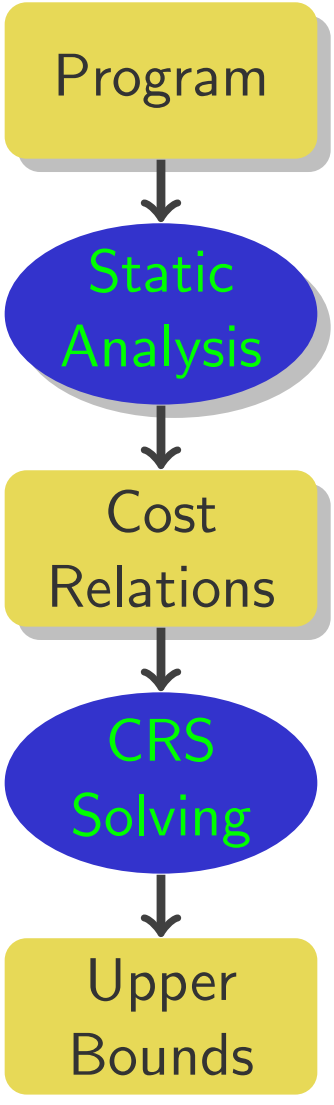


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$$rpop(t,m) = 0 \quad \{m \geq 1\}$$

$$rpop(t,m) = m + rpop(t_1, m) \quad \{m \geq 1, t \geq 1, t_1 = t-1\}$$

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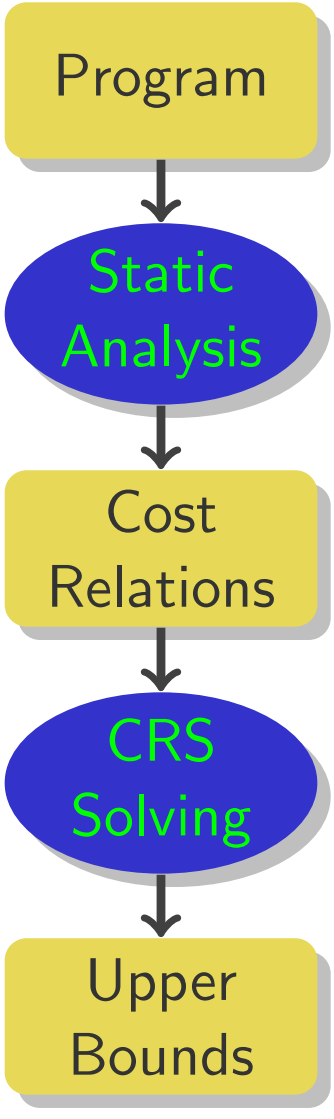


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$$main^u(t,m) = \frac{m^3 + 3t(m^2+m) + 5m}{6} \quad rpop^u(t,m) = m * t$$

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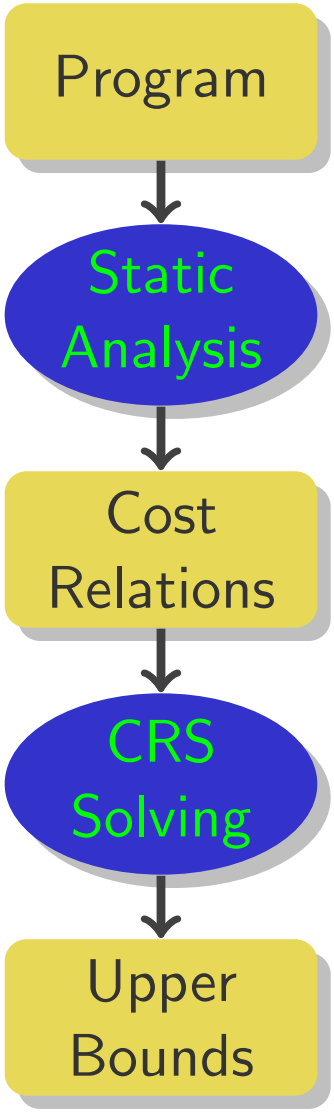


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What fails?

$$main^u(t,m) = \frac{m^3 + 3t(m^2+m) + 5m}{6} \qquad rpop^u(t,m) = m * t$$

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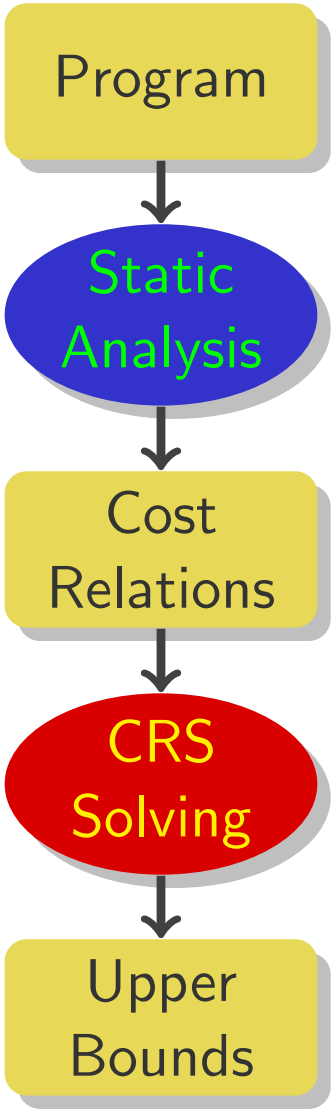

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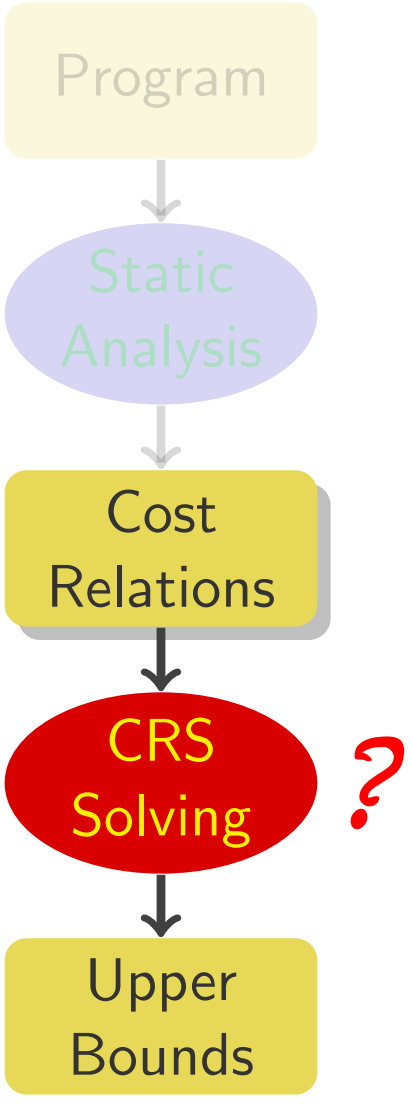

We consider a worst-case evaluation of $main(5, 3)$

$$main^u(t, m) = \frac{m^3 + 3t(m^2 + m) + 5m}{6} \qquad rpop^u(t, m) = m * t$$

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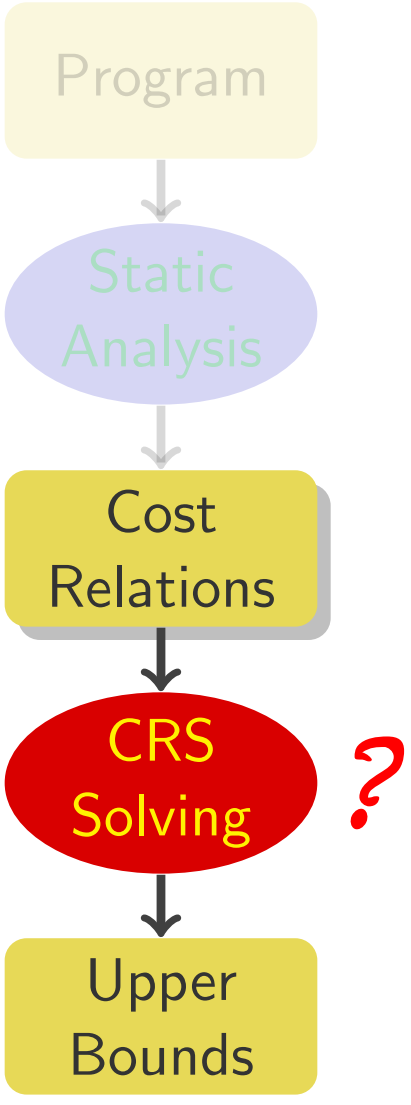
Classical Approach to Cost Analysis: working example



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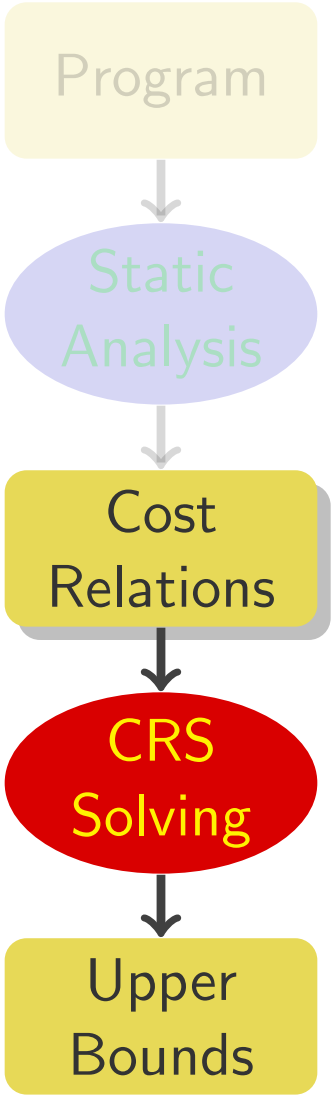

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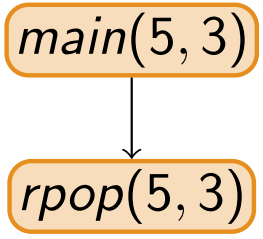

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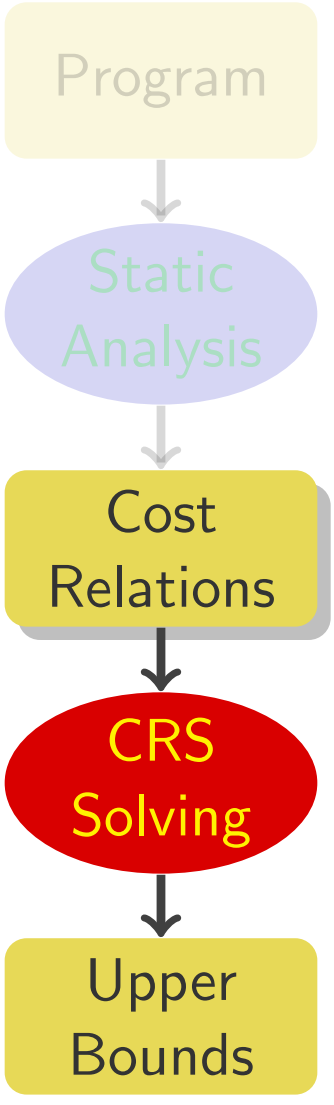

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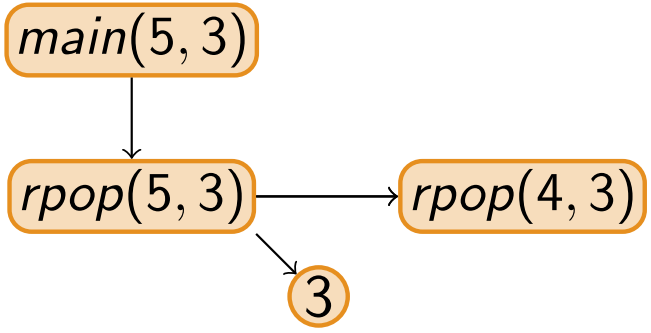

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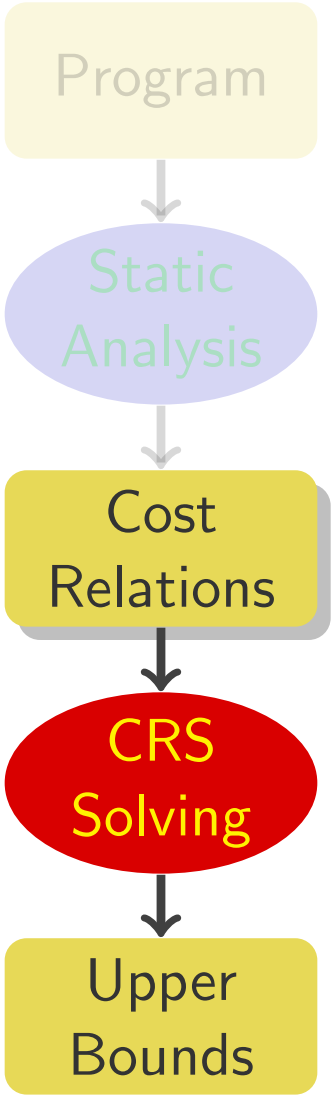

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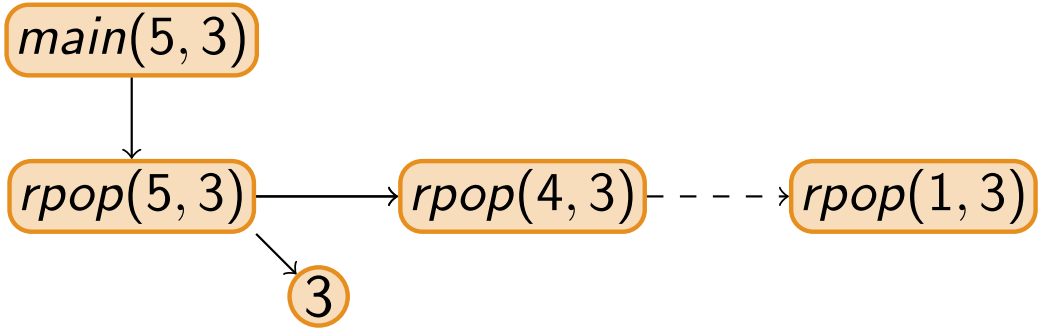

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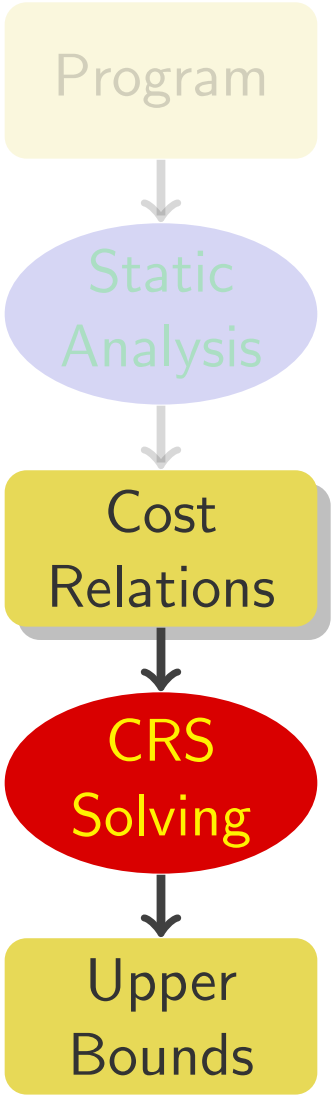

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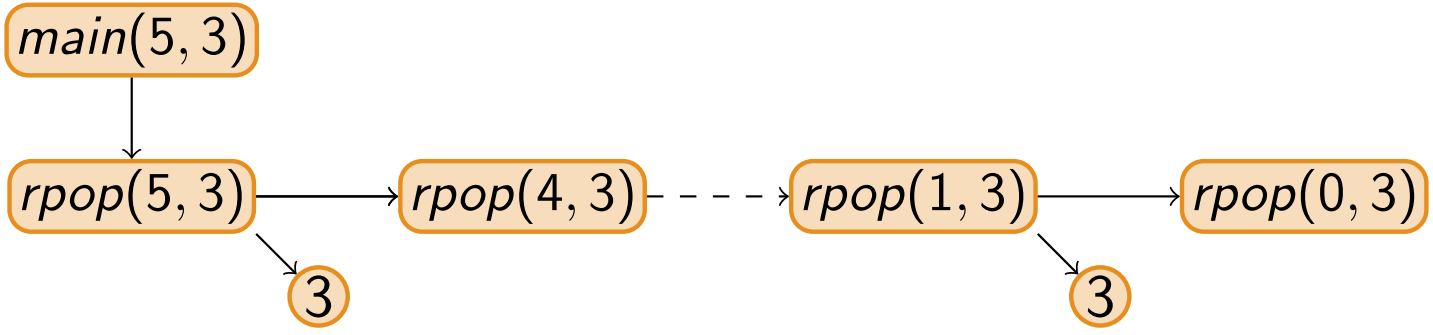

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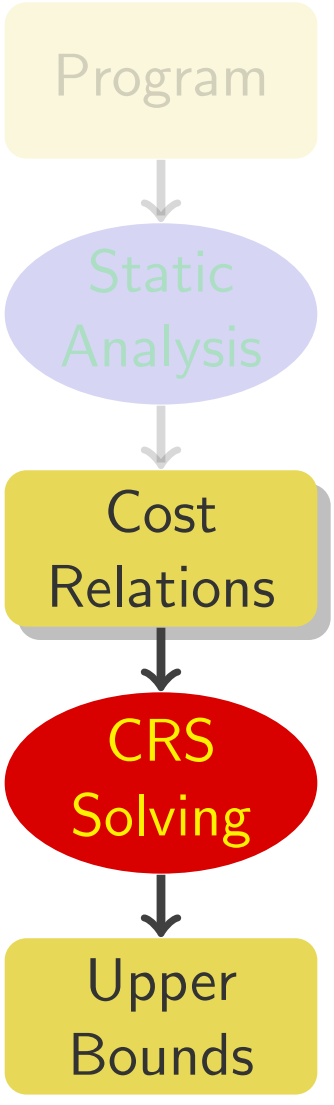

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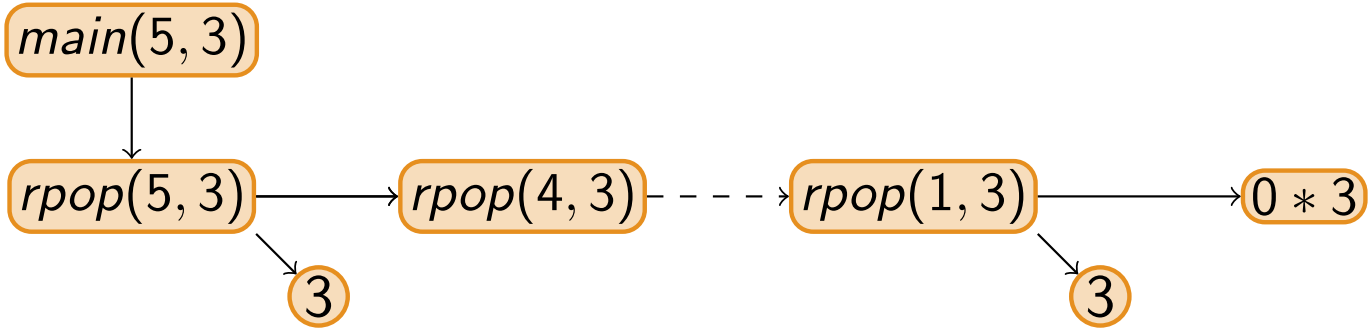

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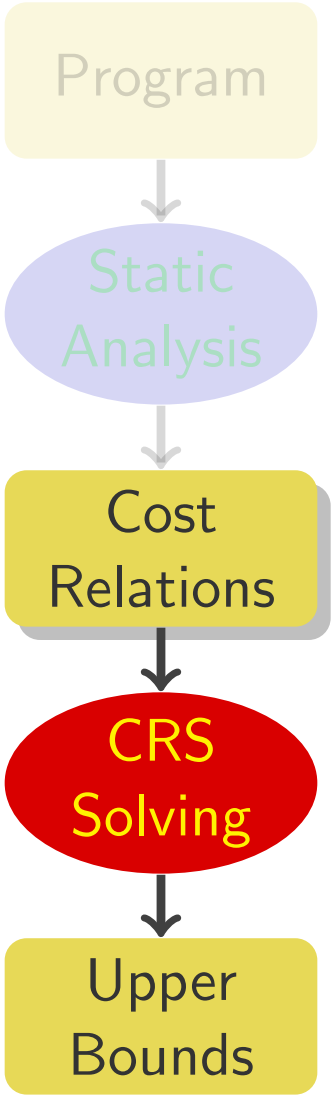

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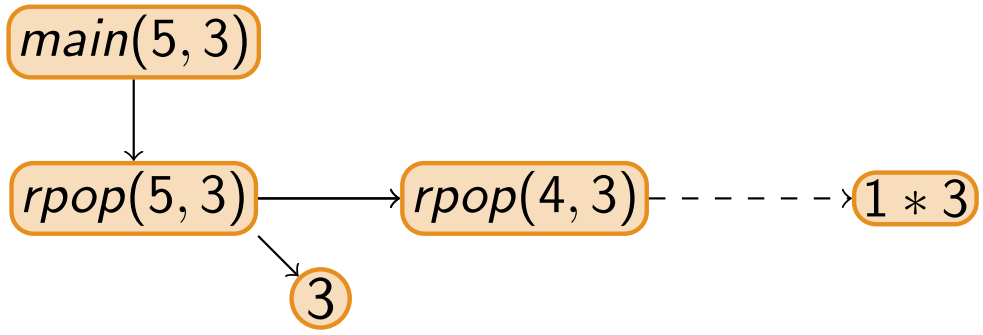

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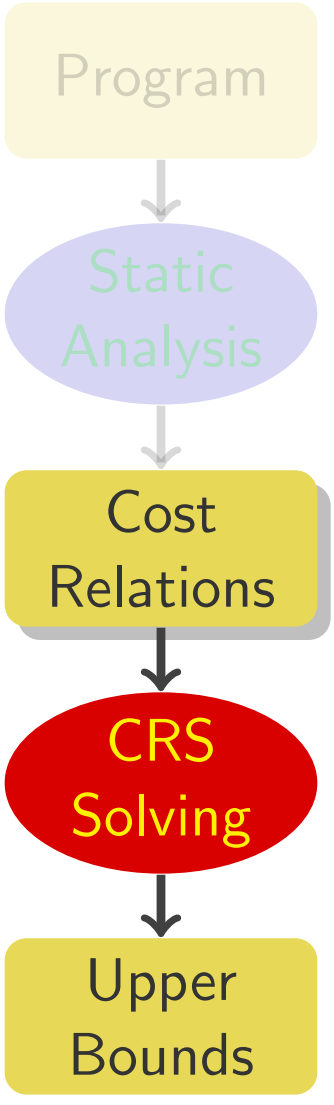

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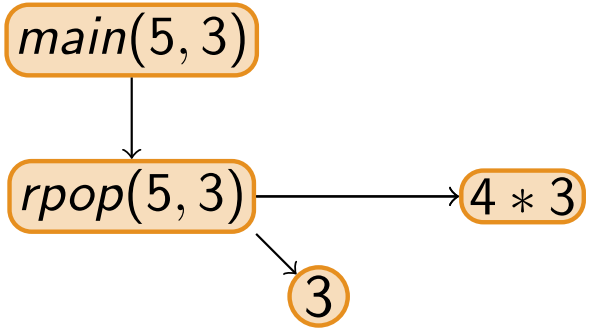

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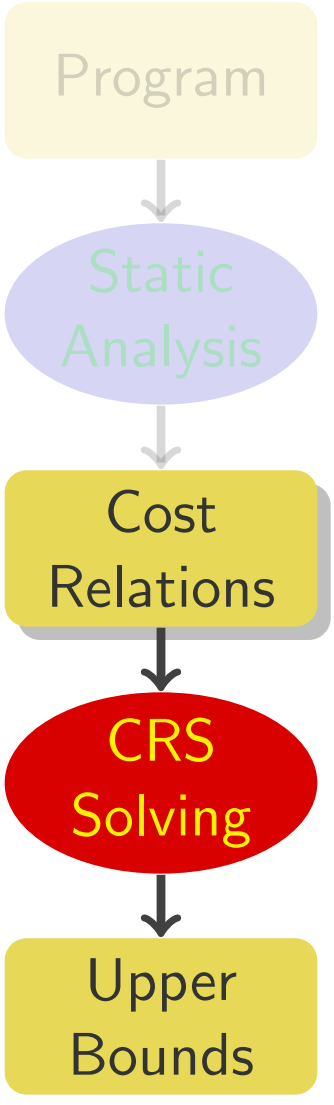

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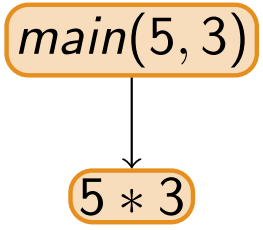

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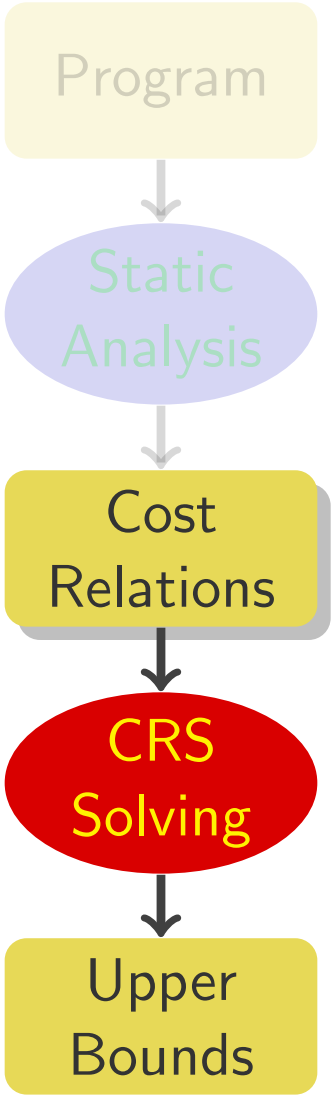

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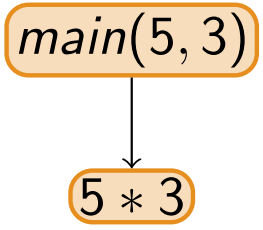

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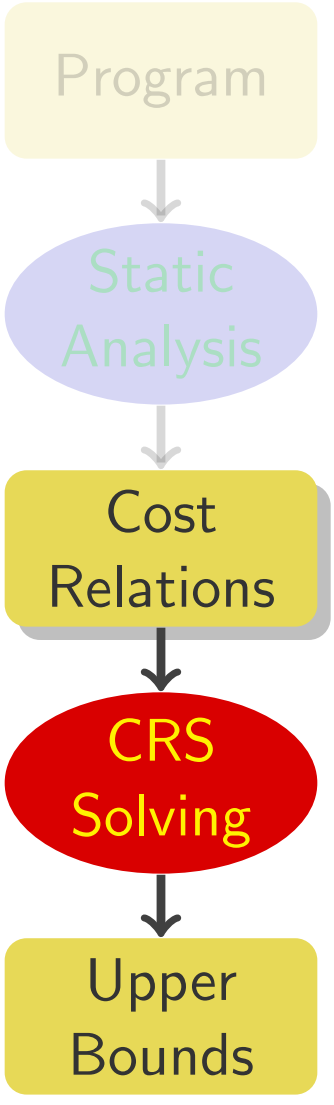

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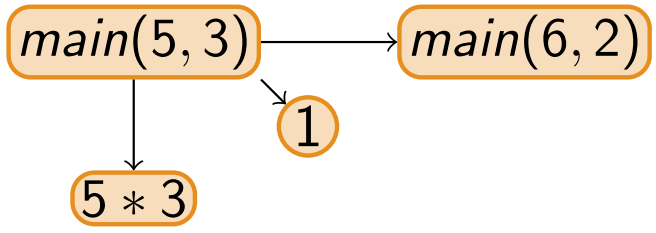

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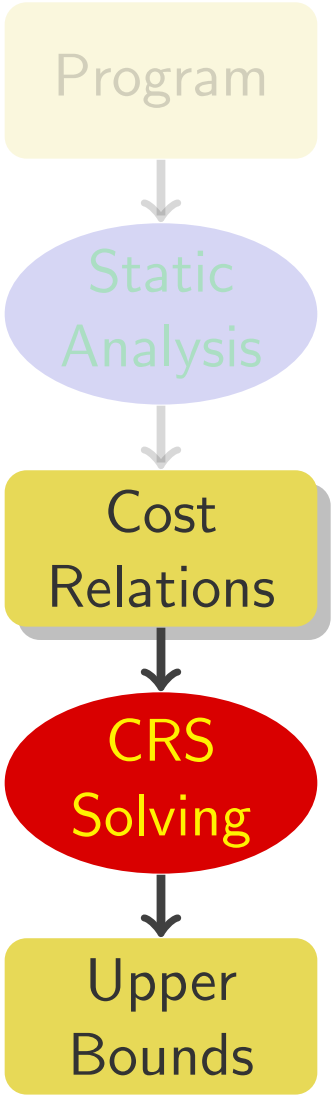

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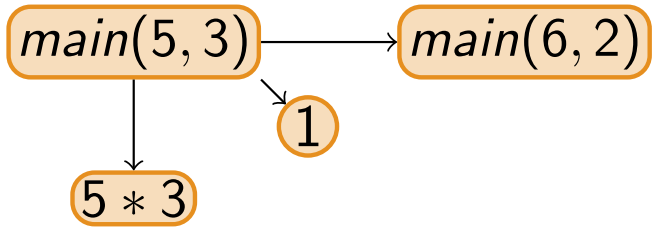

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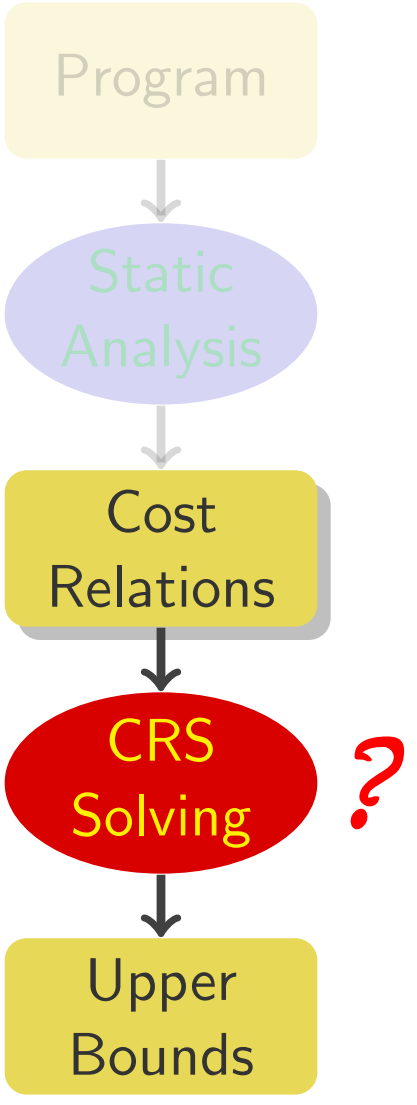

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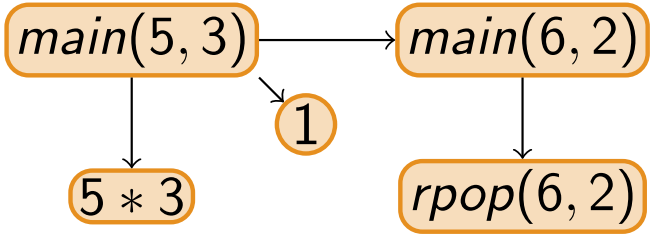

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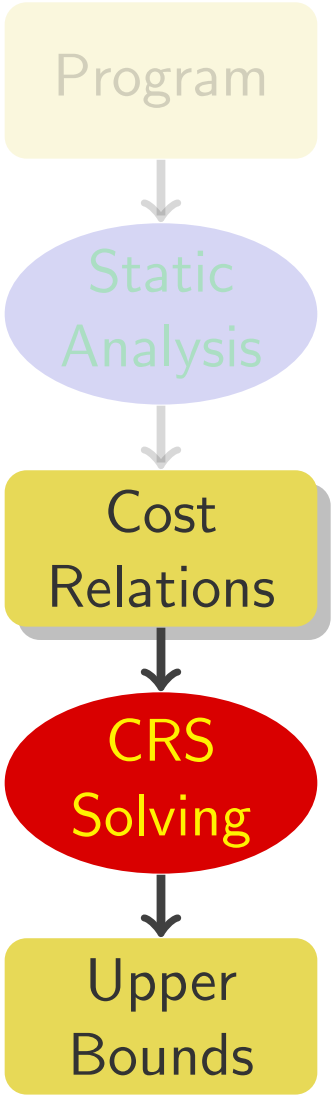

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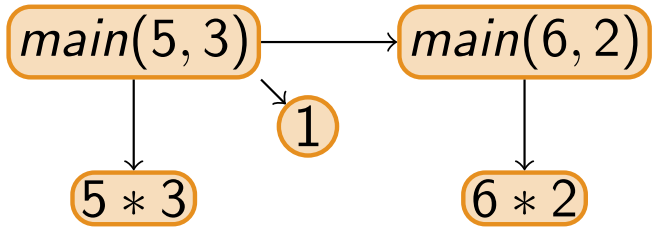

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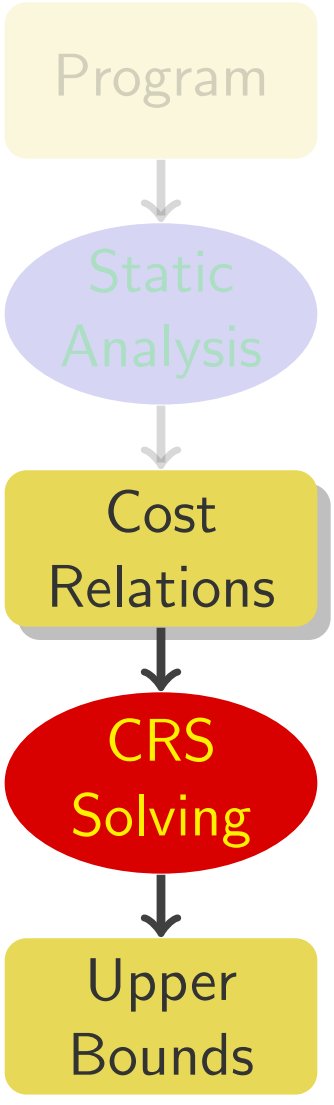

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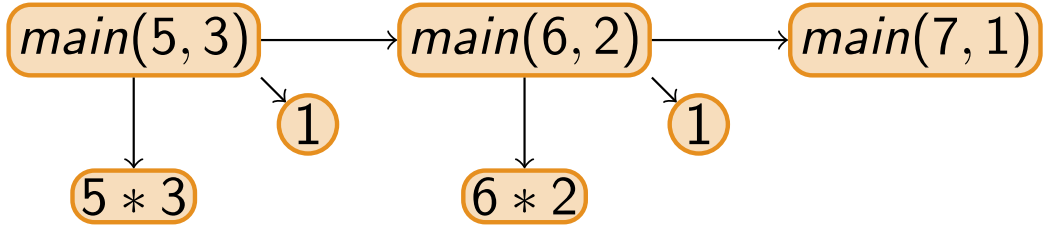

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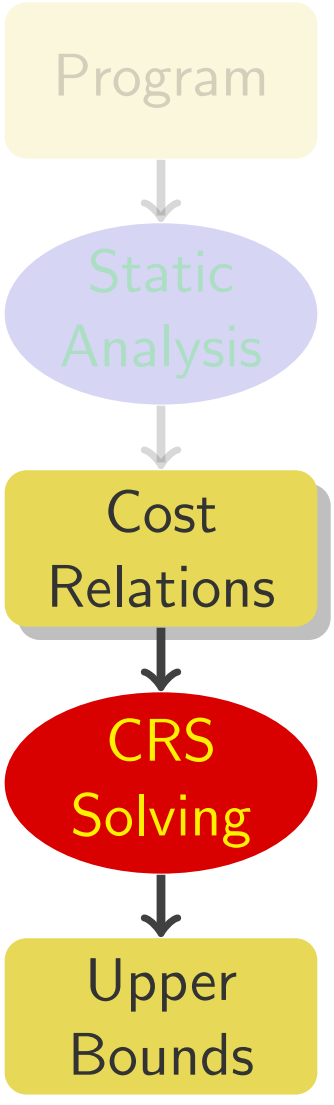

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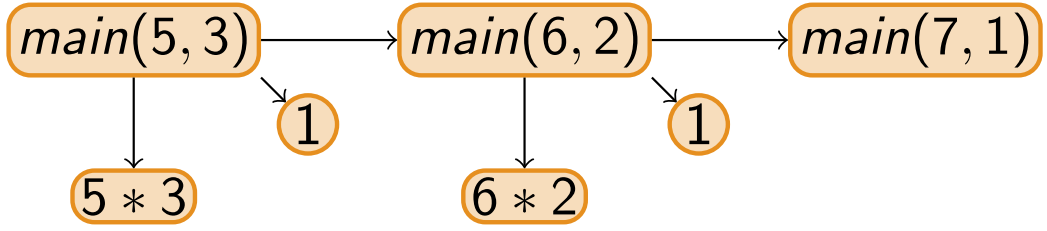
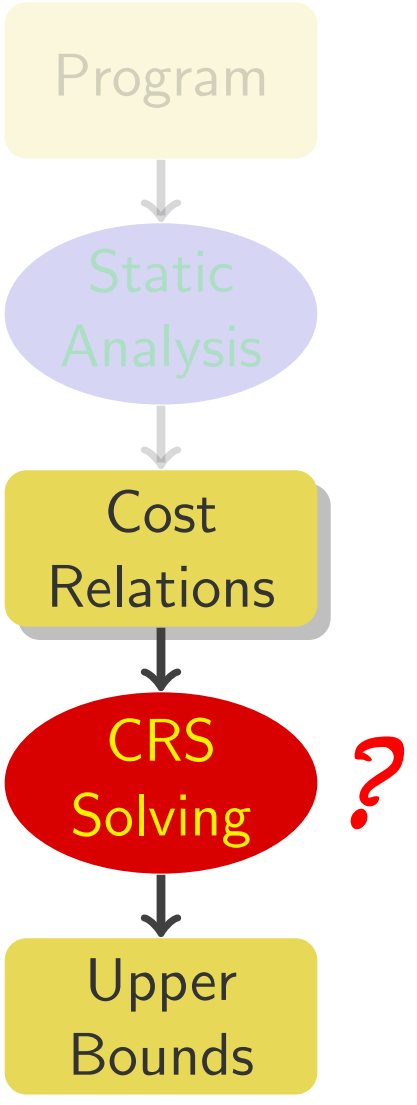


$$main^u(t,m) = \frac{m^3 + 3t(m^2+m) + 5m}{6} \qquad rpop^u(t,m) = m * t$$

$$\begin{array}{lll} main(t,m) = 0 & & \{m = 0, t \geq 0\} \\ main(t,m) = \begin{array}{l} rpop(t,m) + 1 + \\ main(t_2, m_1) \end{array} & & \left\{ \begin{array}{l} m \geq 1, t \geq 0, m_1 = m - 1 \\ t_2 = t' + 1, t \geq t' \geq 0 \end{array} \right\} \\ \\ rpop(t,m) = 0 & & \{m \geq 1\} \\ rpop(t,m) = m + rpop(t_1, m) & & \{m \geq 1, t \geq 1, t_1 = t - 1\} \end{array}$$

Classical Approach to Cost Analysis: working example



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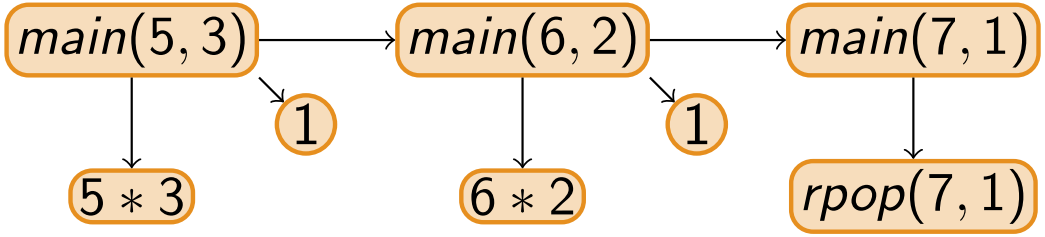

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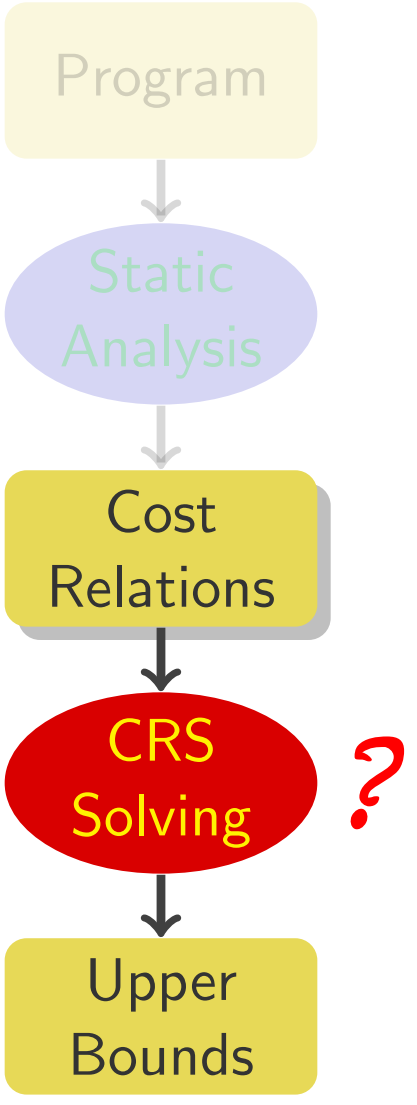
Classical Approach to Cost Analysis: working example



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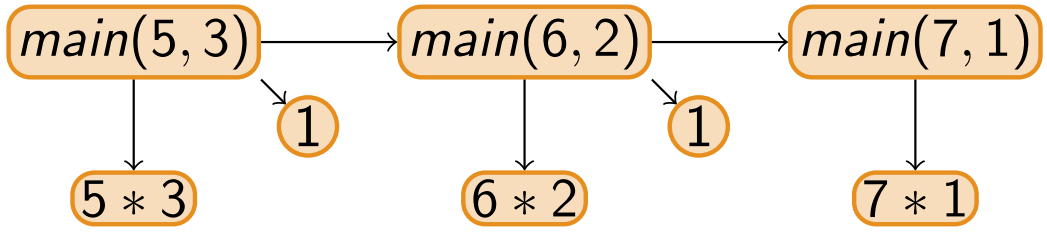

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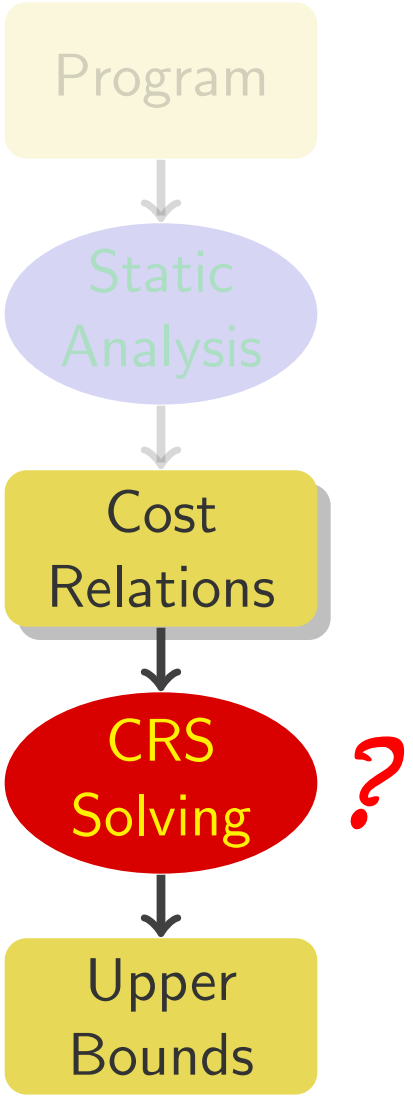

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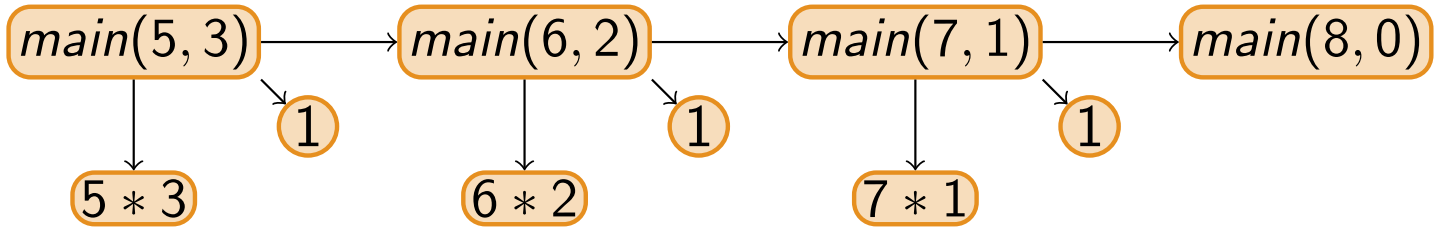

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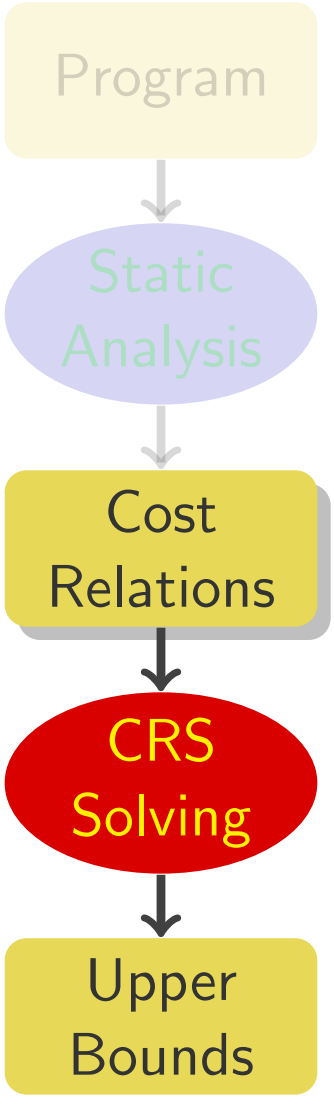
Classical Approach to Cost Analysis: working example



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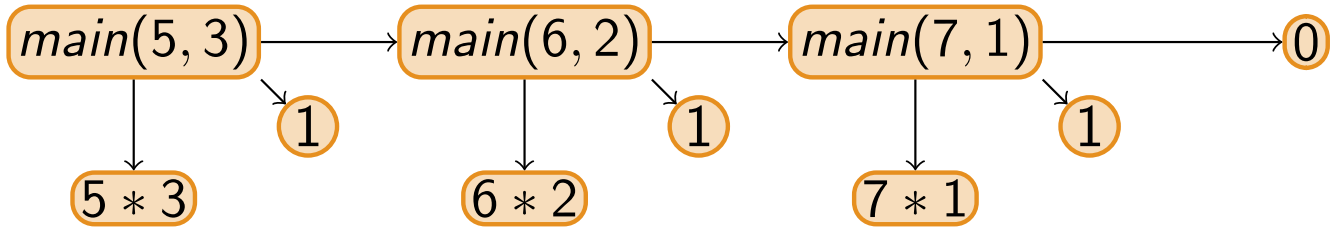

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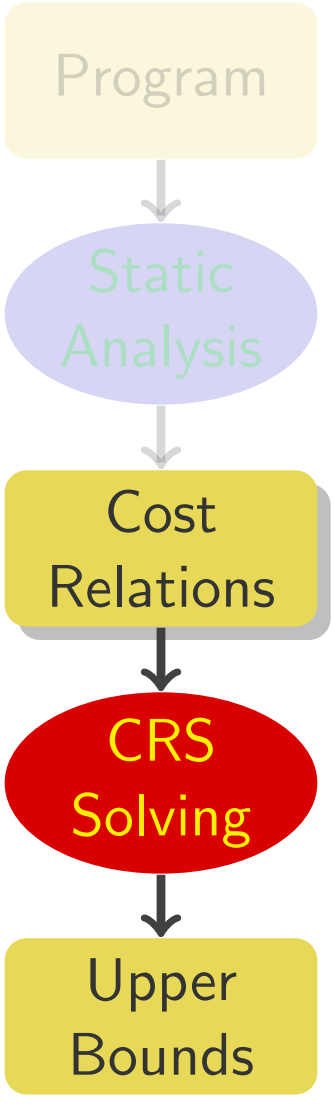

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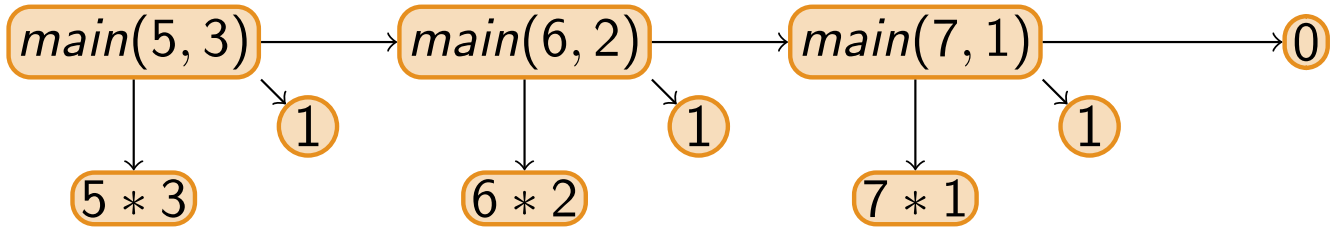

$$main(5,3) = 5*3+1+6*2+1+7*1+1+0 = 37$$

$$main^u(t,m) = \frac{m^3 + 3t(m^2+m) + 5m}{6} \qquad rpop^u(t,m) = m * t$$

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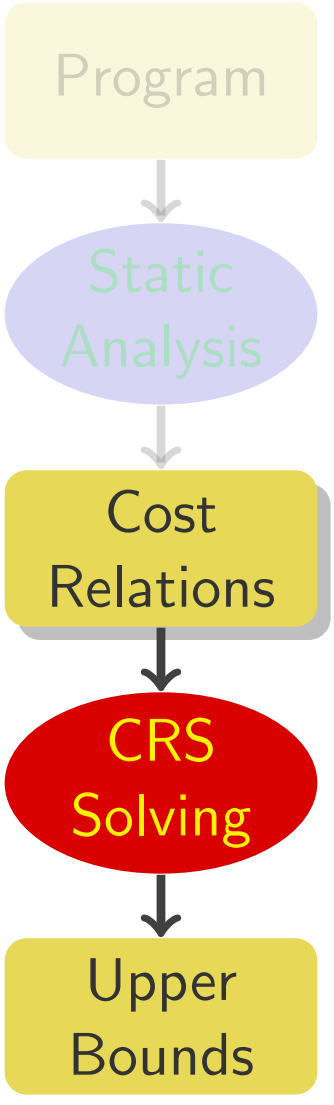

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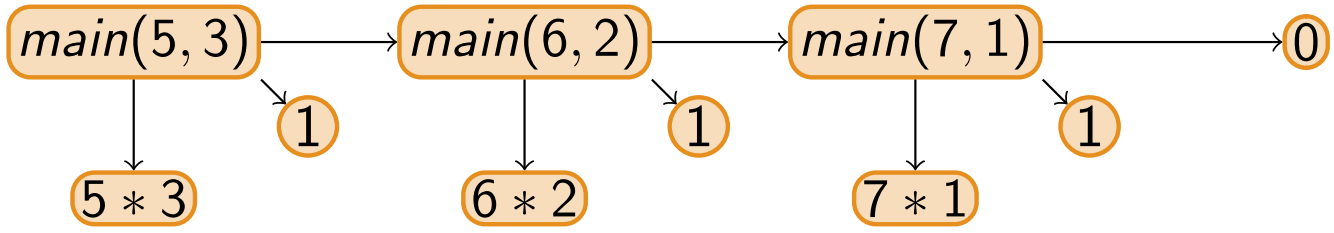

$$main^u(5,3) = \frac{3^3+3*5(3^2+3)+5*3}{6} = 37$$

$$main^u(t,m) = \frac{m^3+3t(m^2+m)+5m}{6} \qquad rpop^u(t,m) = m*t$$

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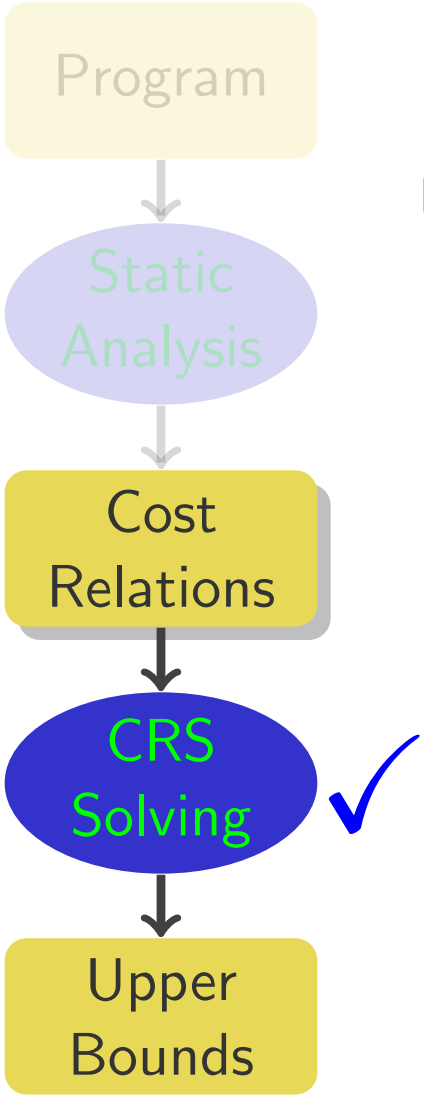
Classical Approach to Cost Analysis: working example



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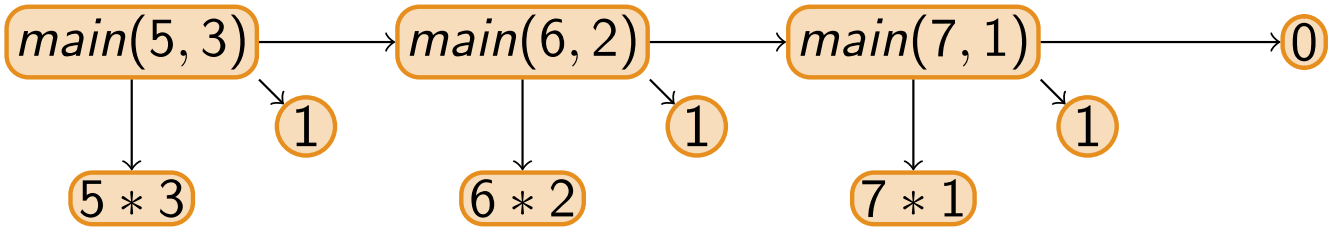

PUBS gives the exact UB

$m) = m * t$

$$main(t,m) = 0 \qquad \{m = 0, t \geq 0\}$$

$$main(t,m) = \begin{matrix} rpop(t,m) + 1 + \\ main(t_2, m_1) \end{matrix} \quad \left\{ \begin{matrix} m \geq 1, t \geq 0, m_1 = m - 1 \\ t_2 = t' + 1, t \geq t' \geq 0 \end{matrix} \right\}$$

$$rpop(t,m) = 0 \qquad \{m \geq 1\}$$

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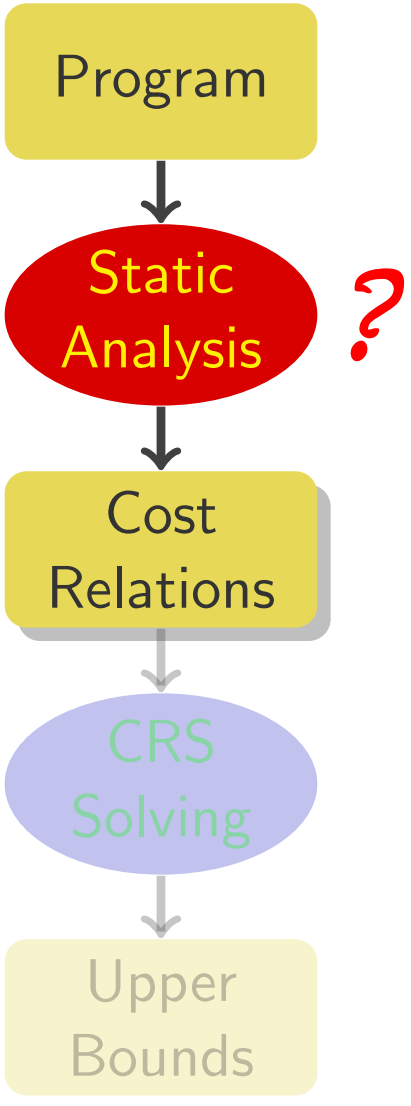
Classical Approach to Cost Analysis: working example



$$
\begin{array}{lll}
main(t,m) = & 0 & \{m = 0, t \geq 0\} \\
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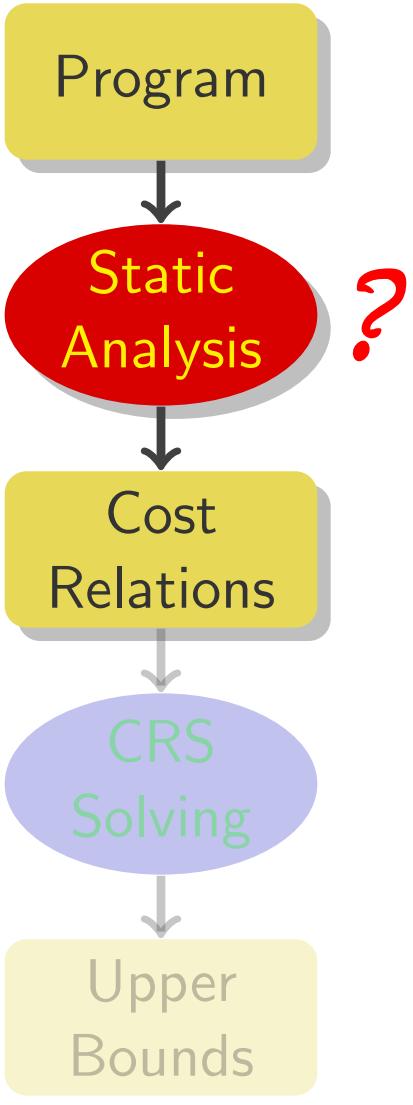
Classical Approach to Cost Analysis: working example




```
//@requires m >= 1
void rpop(int m){
  while(top!=null && *){
    //pop
    top=top.next ;
    //@acquire(m)
}}
```

```
//@requires m >= 0
void main(int m) {
  while( m > 0 ) {
    rpop(m)  ;
    // push(*);
    top = new
     Node(*,top);
    //@acquire(1)
    m = m-1      ;
}}}
```

$$
\begin{array}{ll}
main(t,m) = 0 & \{m = 0, t \geq 0\} \\
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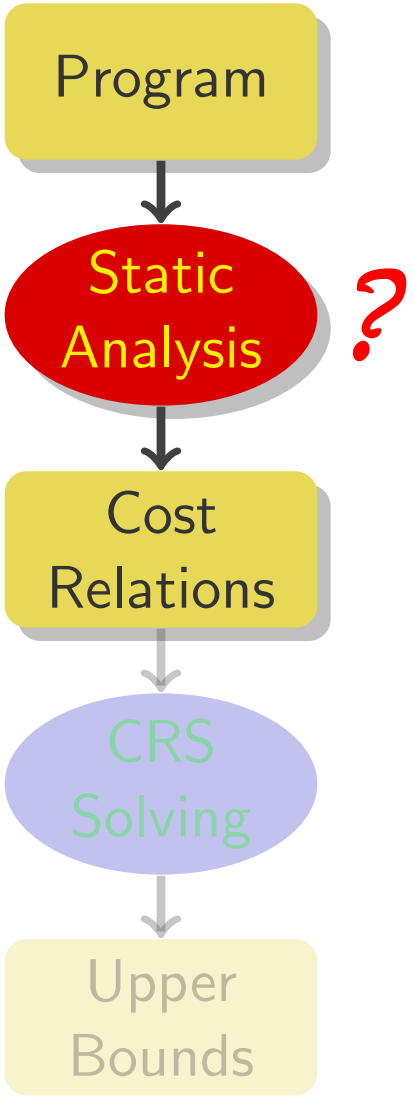


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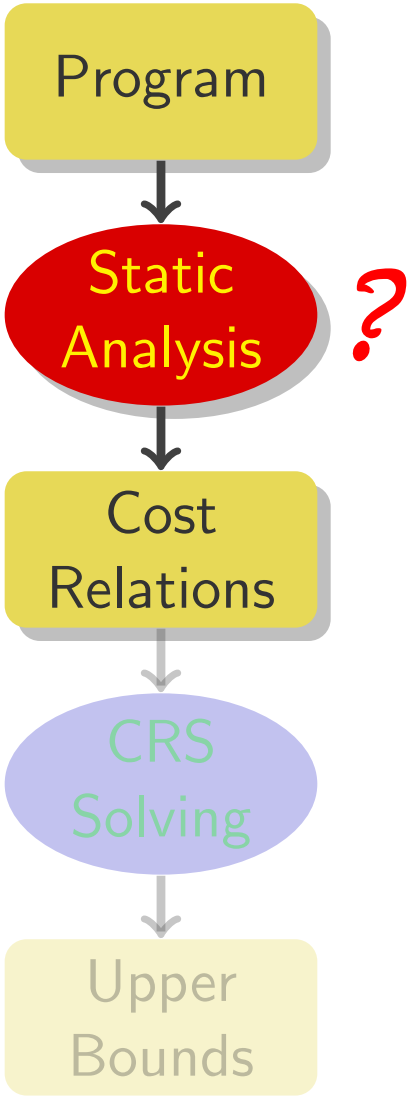


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$s \geq s' \geq 0$ is the exact `rpop` size postcondition

$$
\begin{array}{lll}
main(t,m) = 0 & & \{m = 0, t \geq 0\} \\
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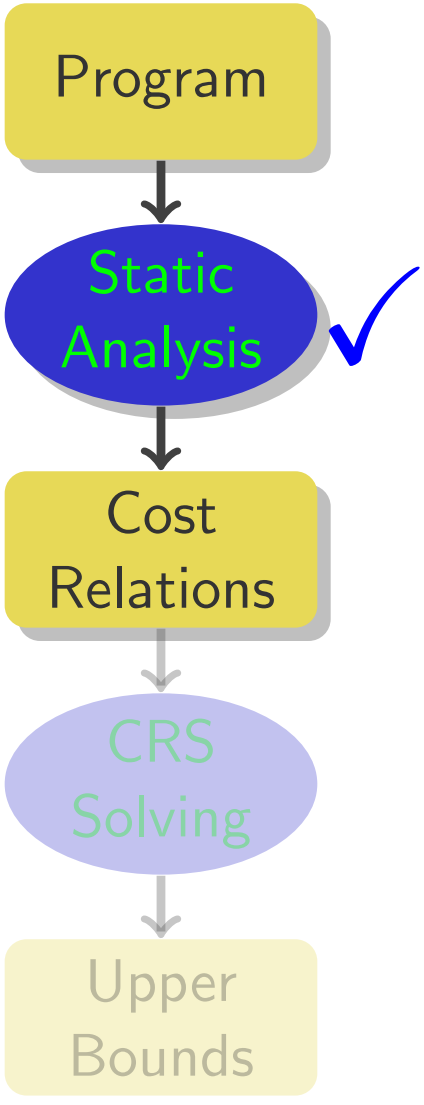
Classical Approach to Cost Analysis: working example

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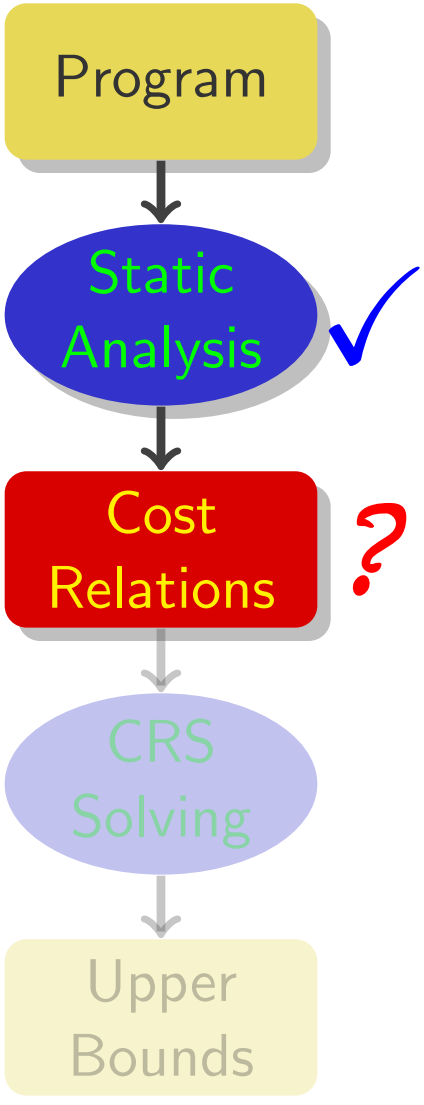
Classical Approach to Cost Analysis: working example

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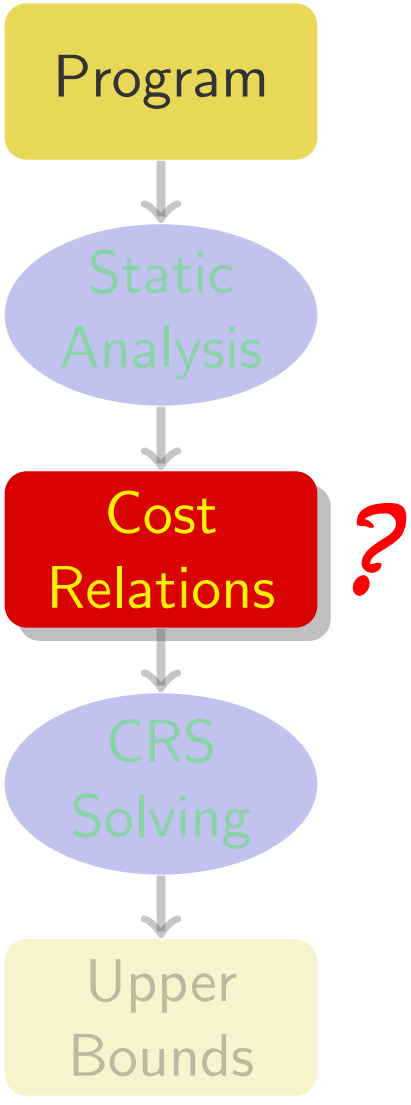

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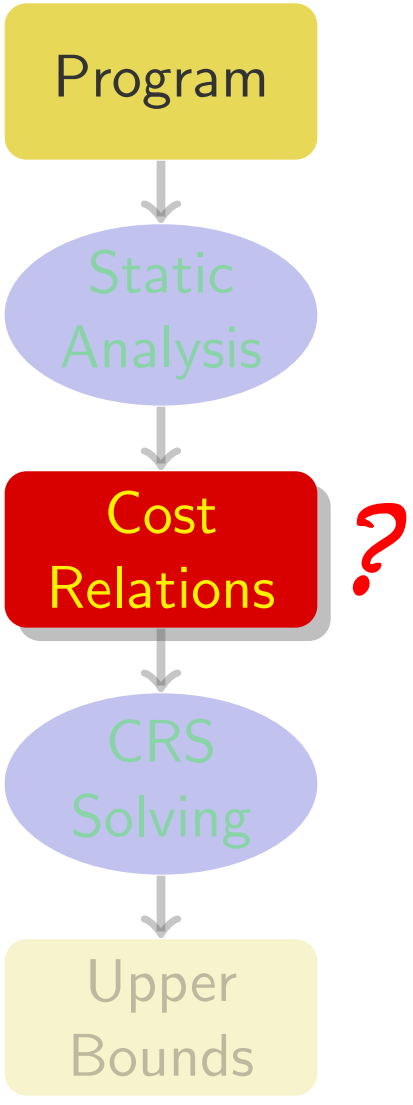

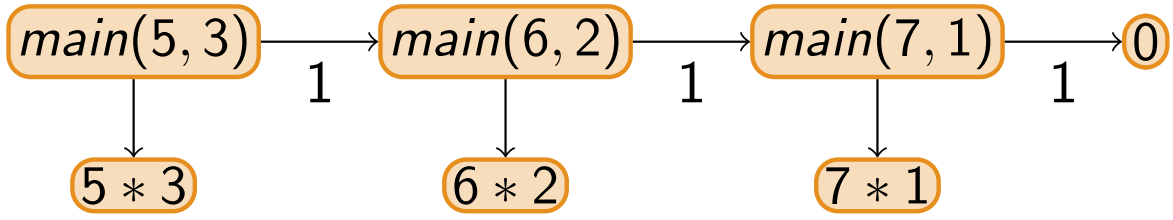

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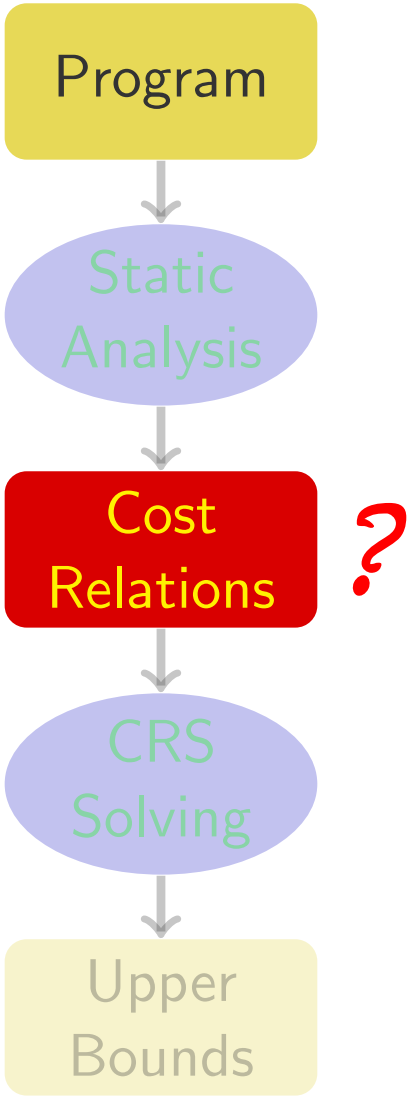

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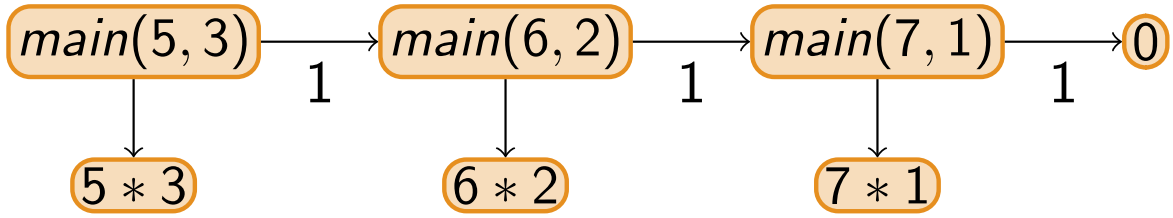
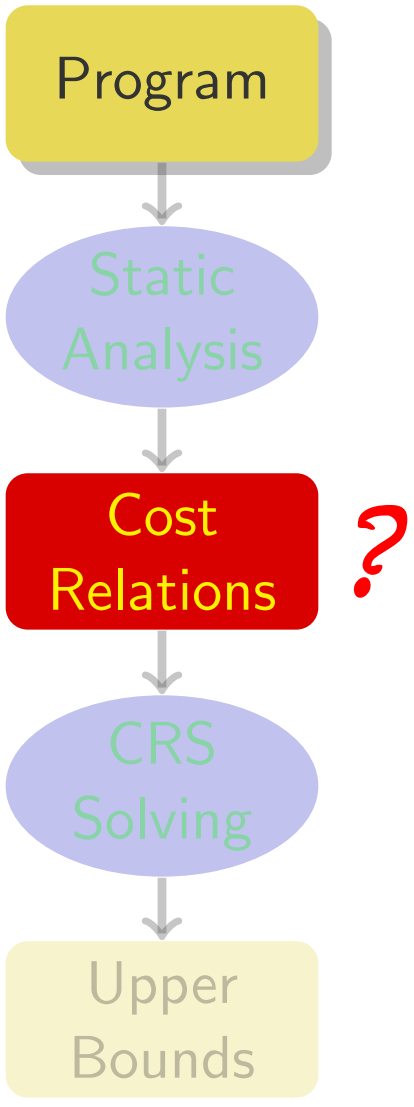


Classical Approach to Cost Analysis: working example



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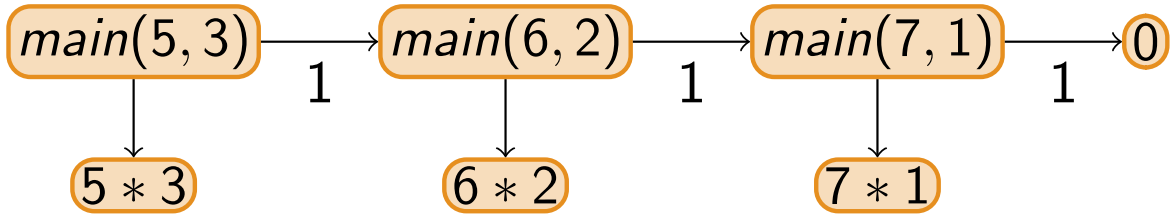


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//@requires m >= 0
void main(int m) {
  while( m > 0 ) {
    rpop(m) ;
    top=new Node(*,top);
    //@acquire(1)
    m = m-1      ; }}}
```

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//@requires m >= 1
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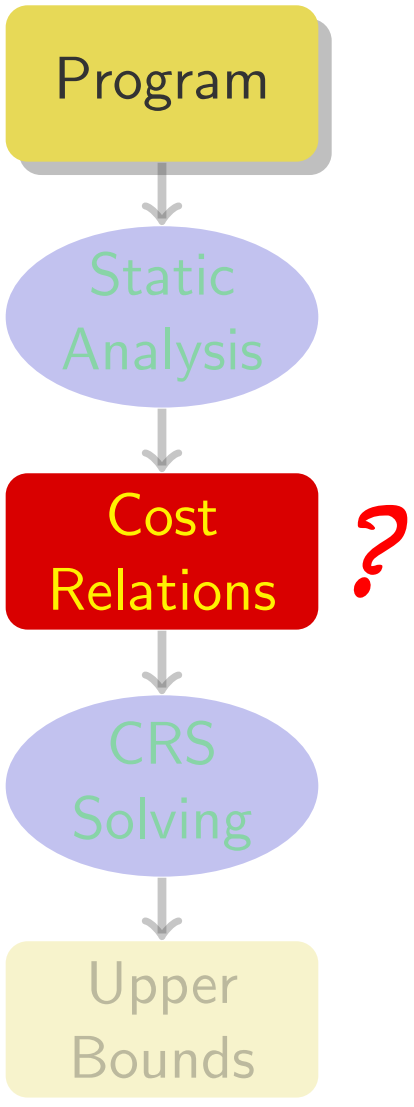
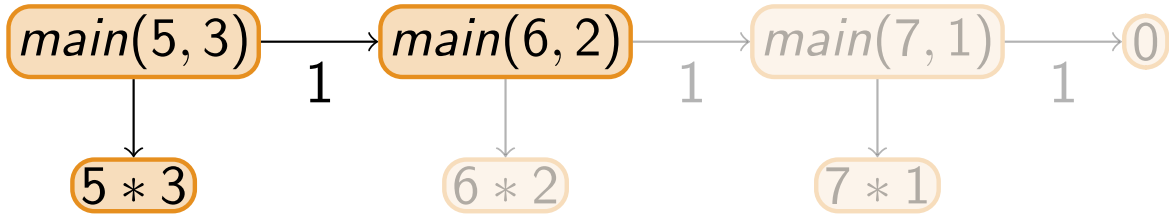
}}
```

while

Classical Approach to Cost Analysis: working example



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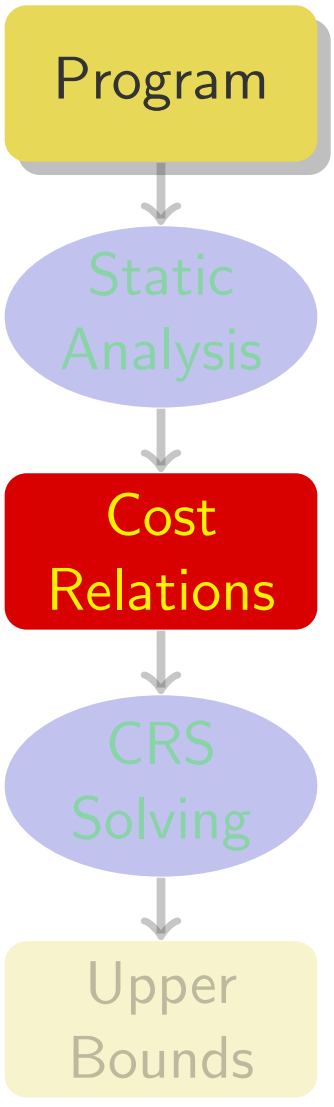
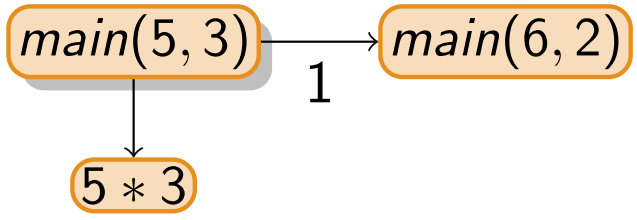
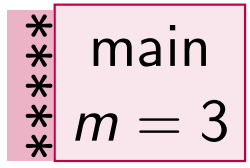
Classical Approach to Cost Analysis: working example







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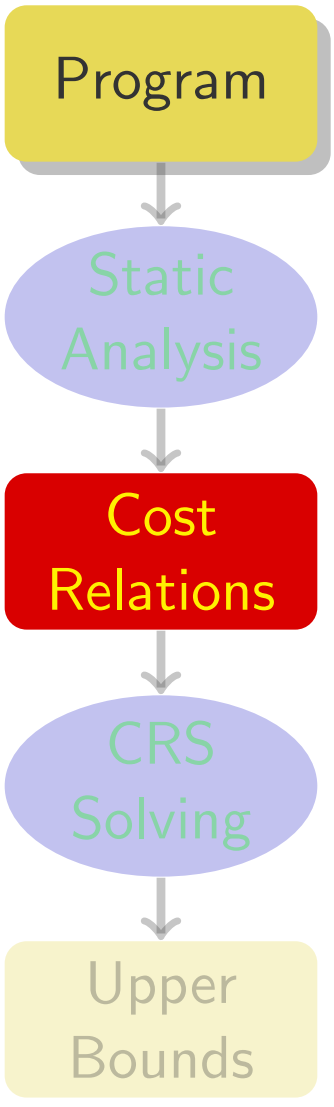
Classical Approach to Cost Analysis: working example



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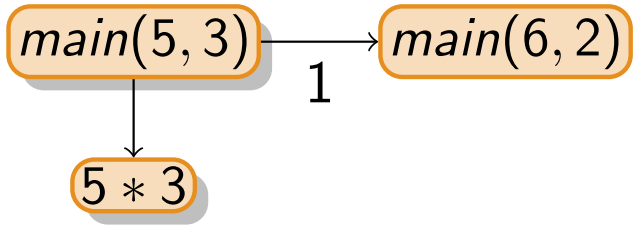

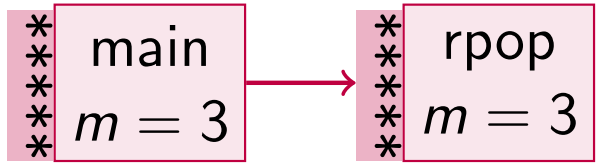


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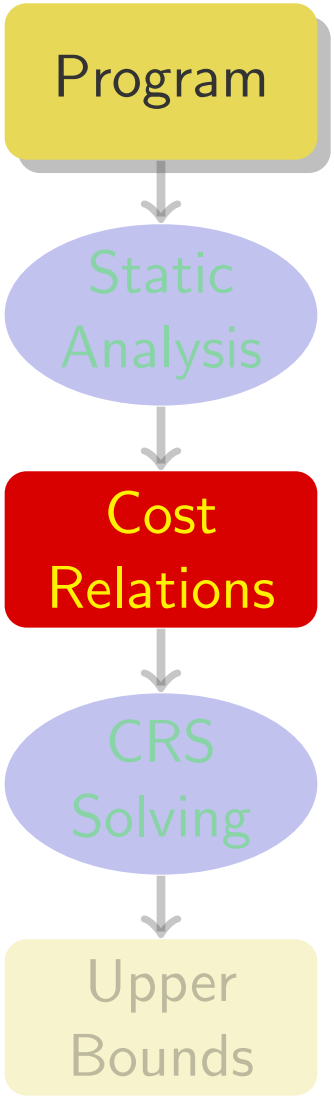
while

Classical Approach to Cost Analysis: working example



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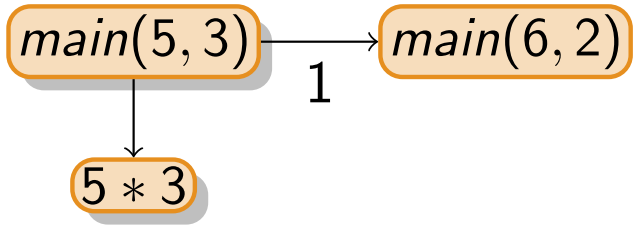

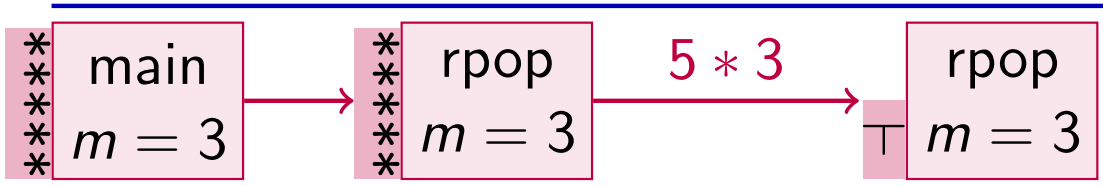


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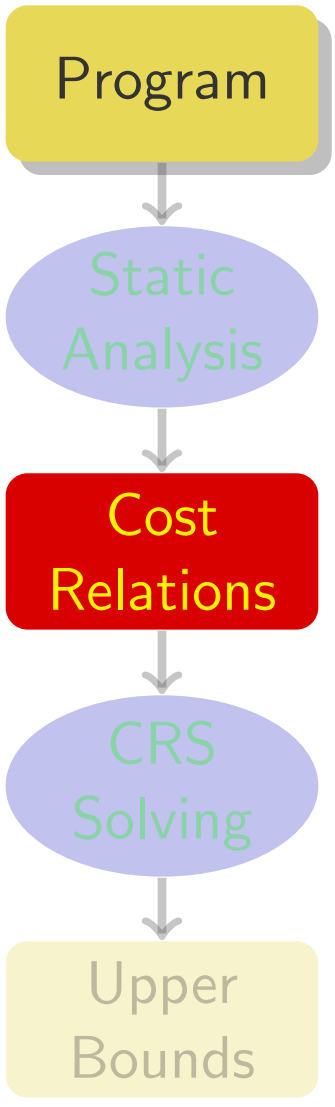
Classical Approach to Cost Analysis: working example



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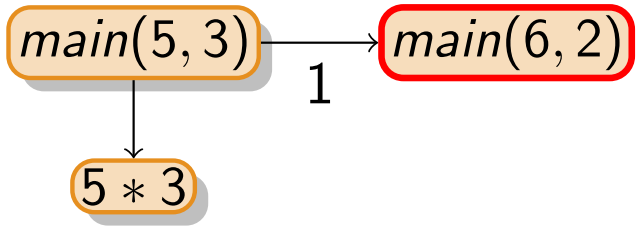

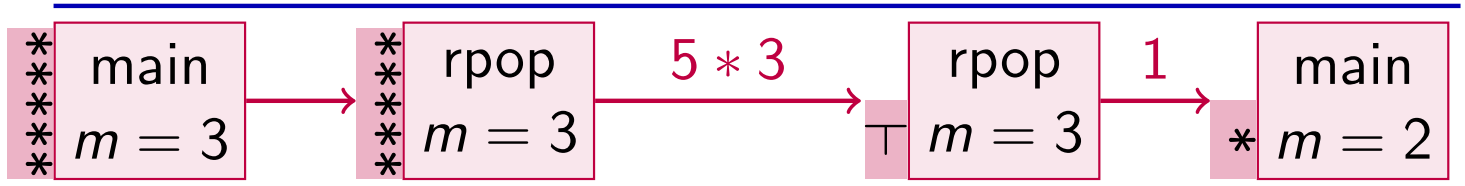


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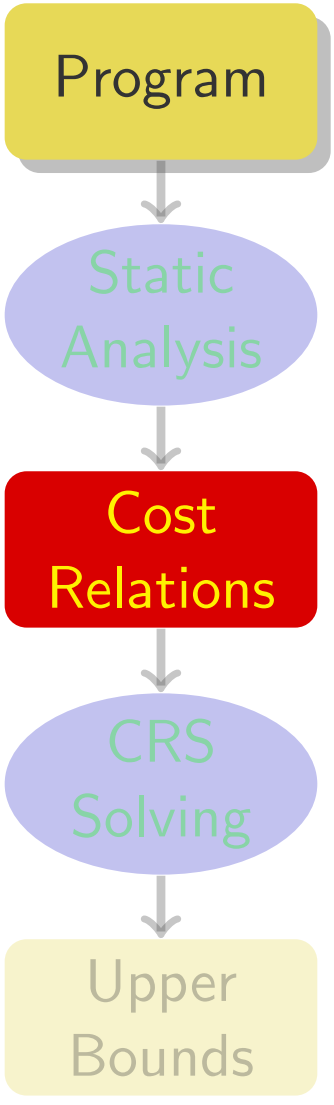
Classical Approach to Cost Analysis: working example



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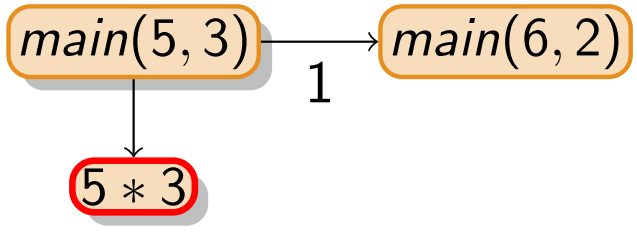

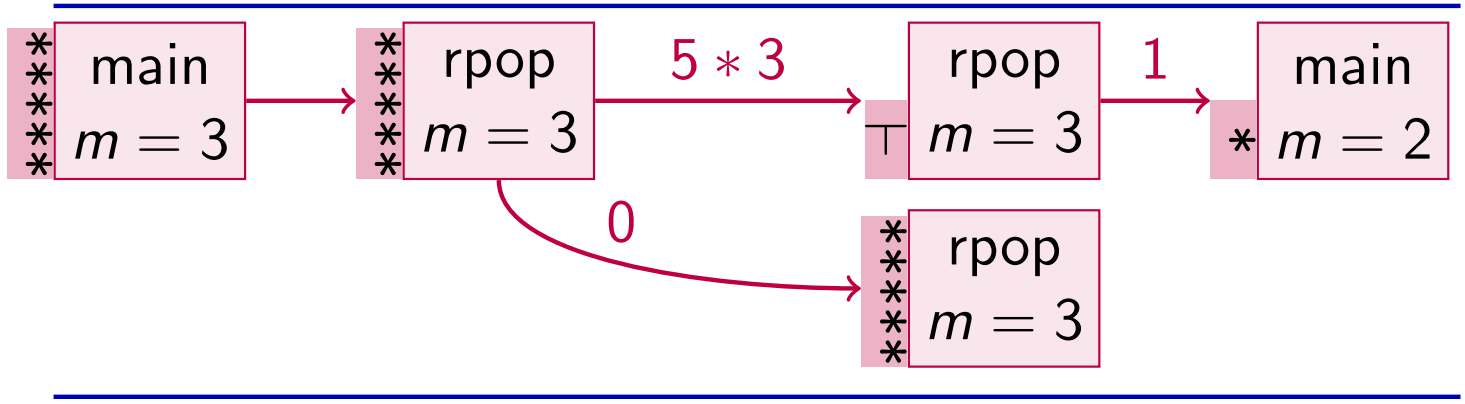


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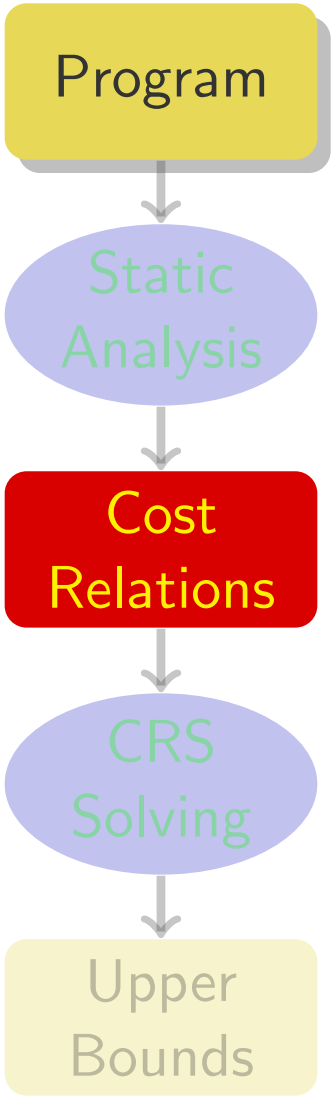
Classical Approach to Cost Analysis: working example



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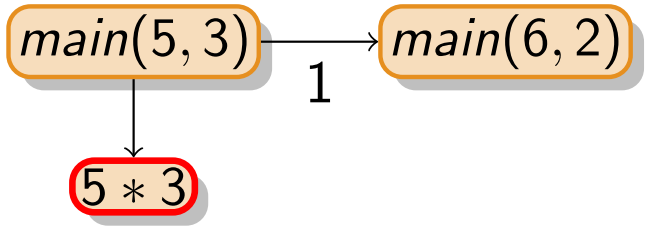

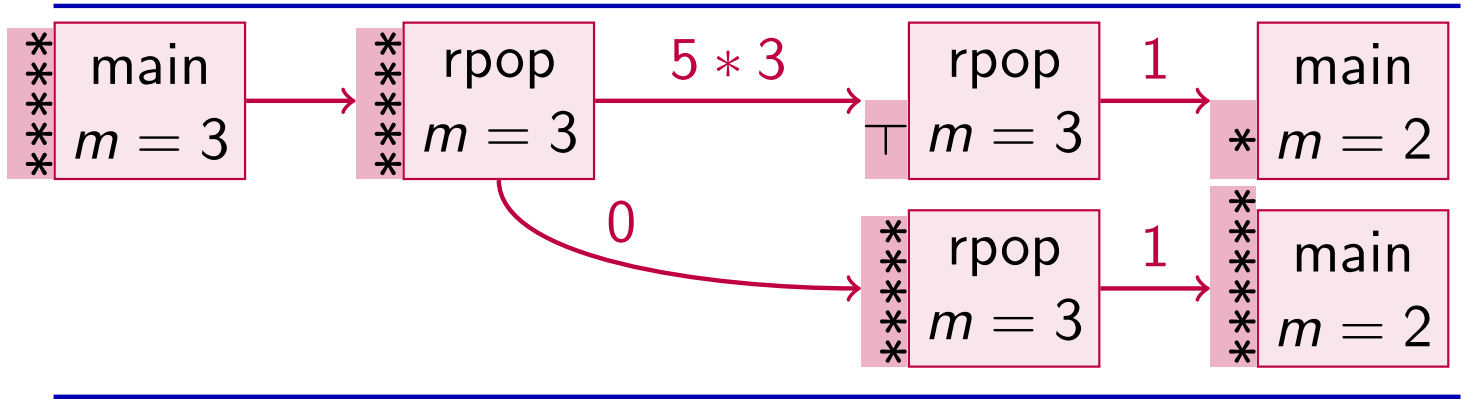


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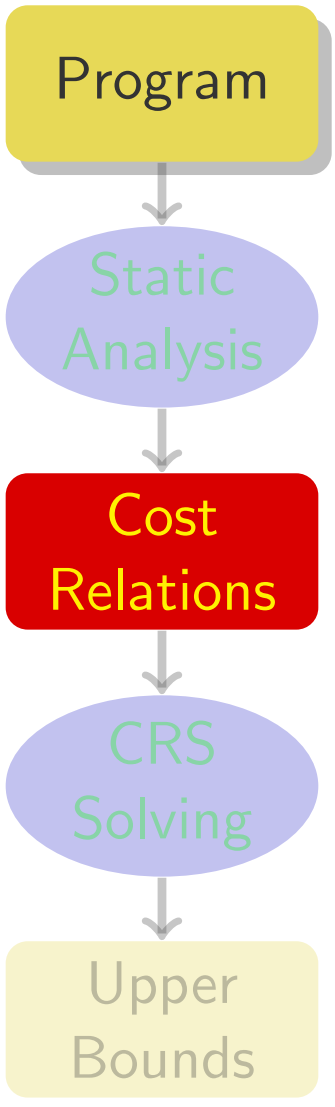
Classical Approach to Cost Analysis: working example



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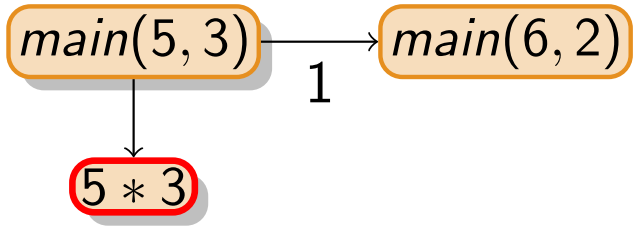
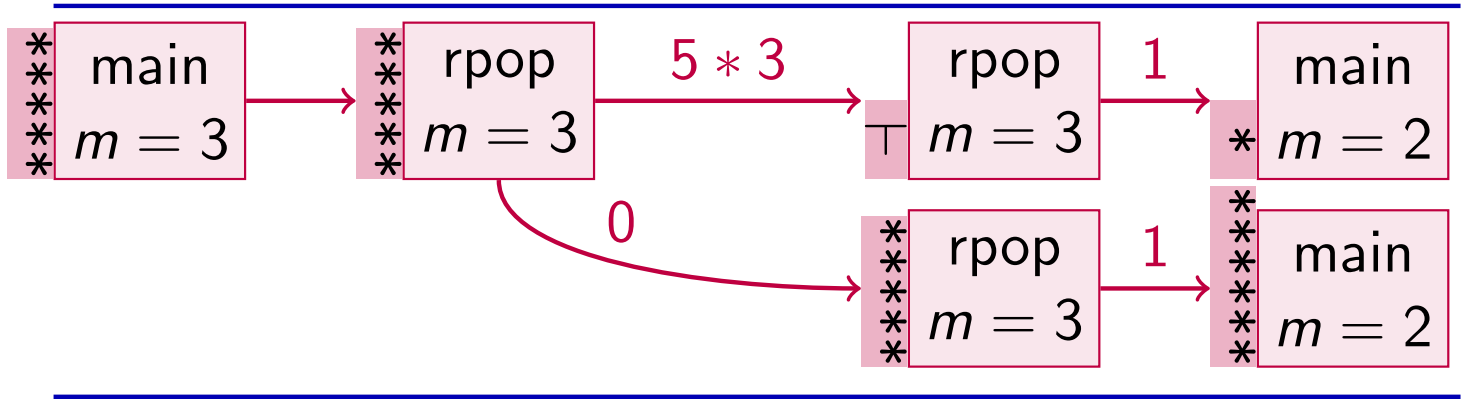


This CR evaluation matches no program execution

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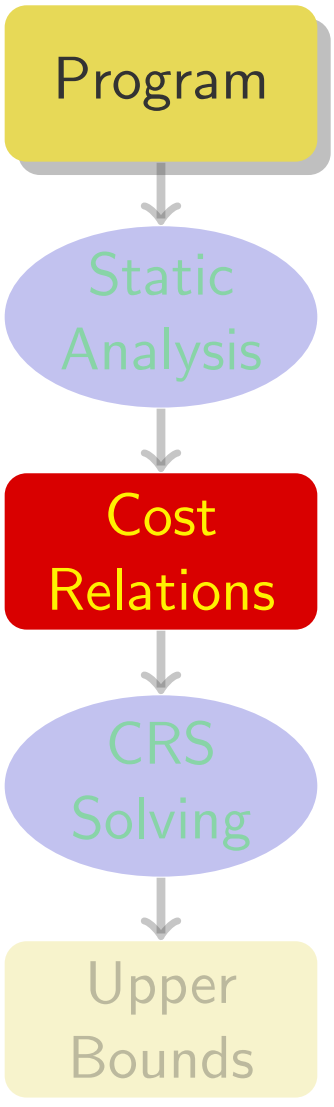
Classical Approach to Cost Analysis: working example



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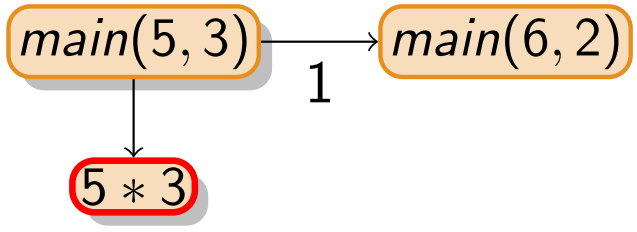
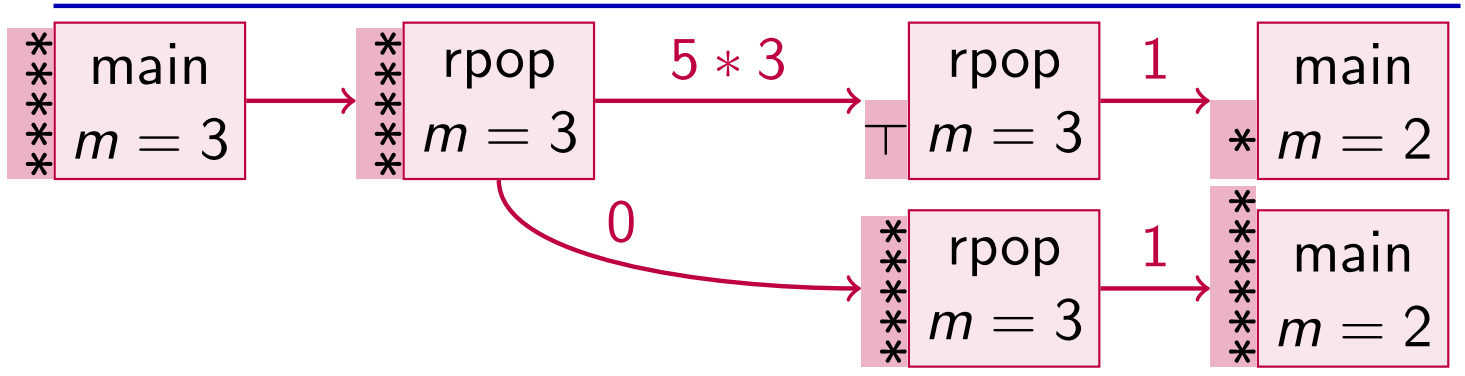


The cost of *rpop* depends on the value of its output

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while

The output-cost correlation

The cost of *rpop* depends on the value of its output

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//@requires m >= 1
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The output-cost correlation

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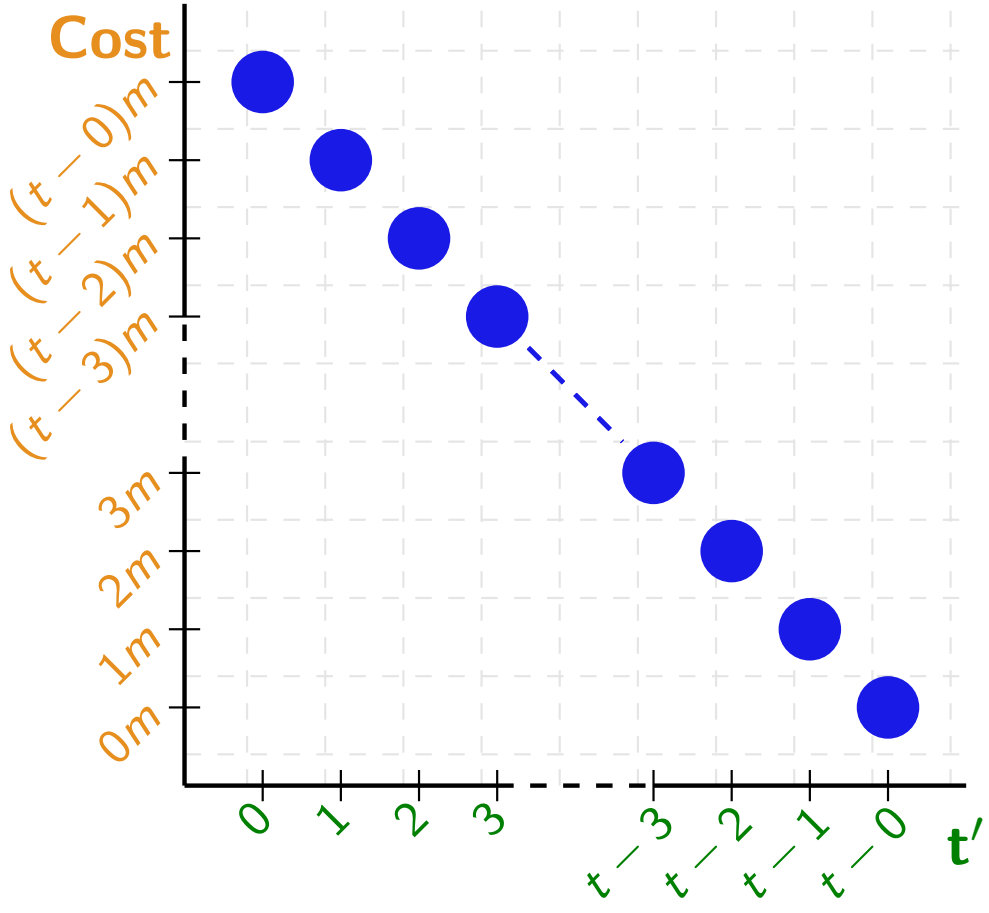


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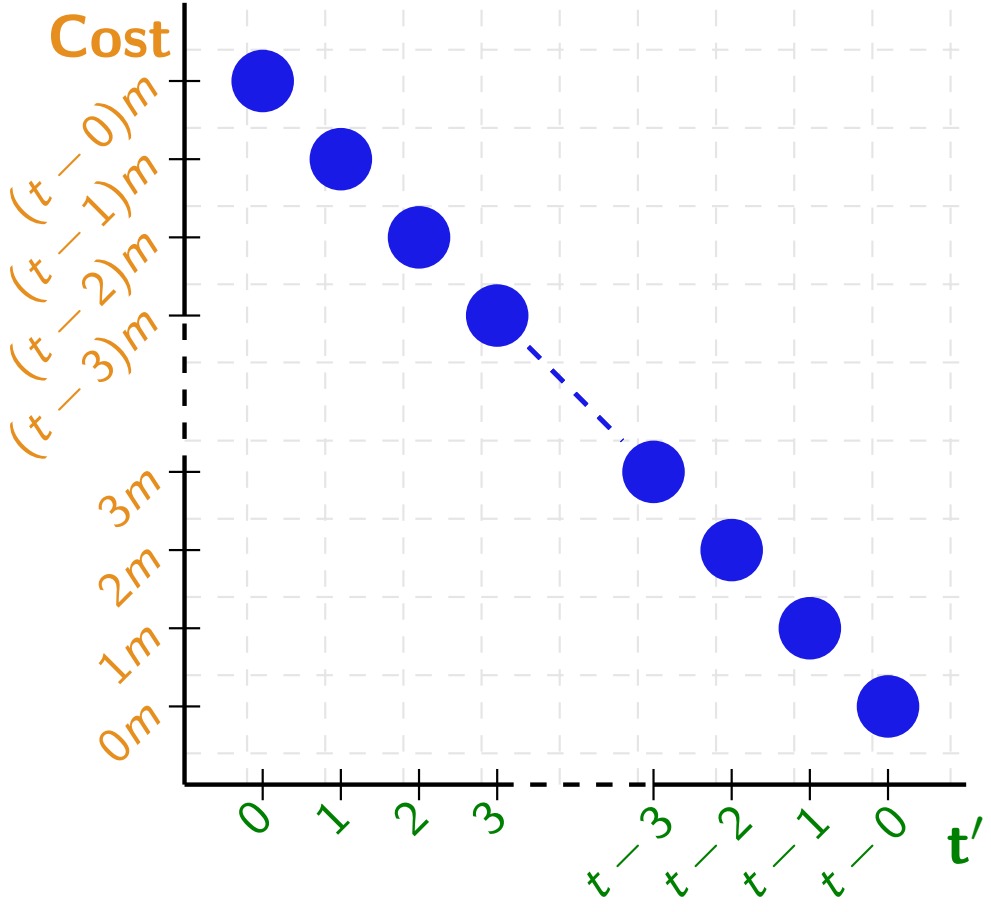
The output-cost correlation

$$main(t, m) = 0 \qquad \{m = 0, t \geq 0\}$$

$$main(t, m) = \begin{matrix} rpop(t, m) + 1 + \\ main(t' + 1, m - 1) \end{matrix} \quad \left\{ \begin{matrix} m \geq 1, t \geq 0 \\ t \geq t' \geq 0 \end{matrix} \right\}$$



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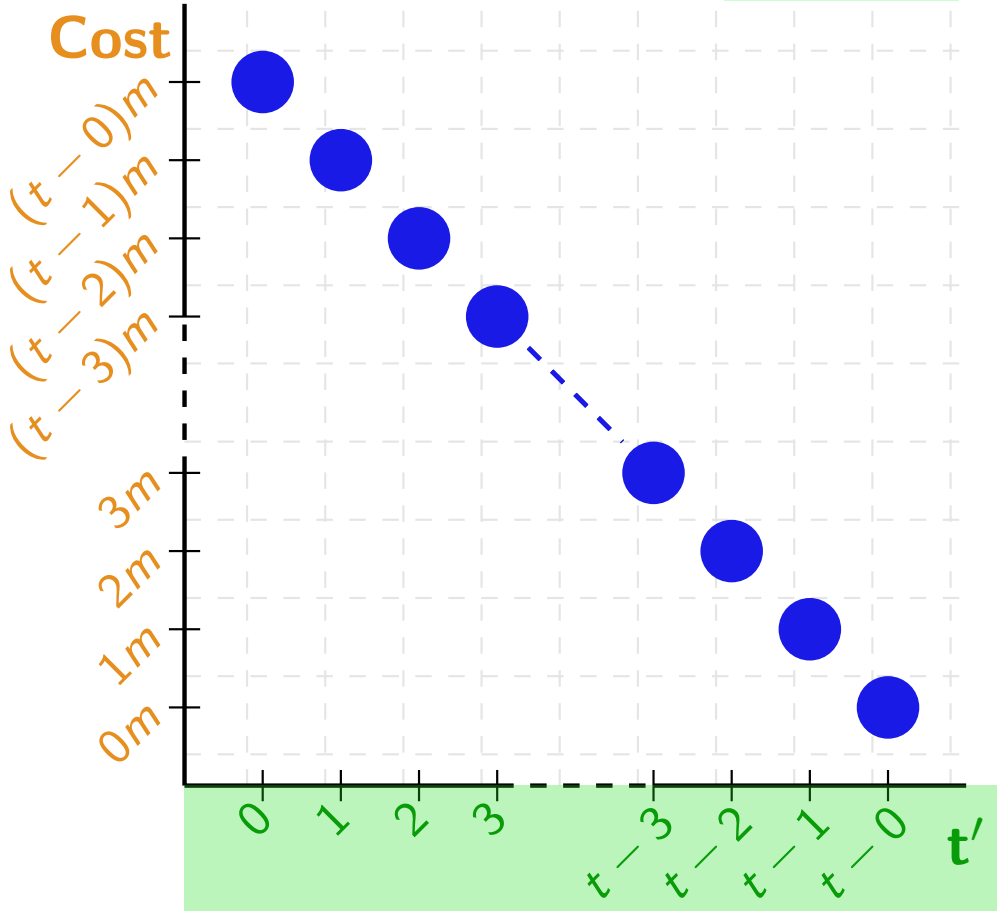

The cost of *rpop* depends on the value of its output

- $t \geq t' \geq 0$ is a precise postcondition

```
//@requires m >= 1
void rpop(int m){
  if/while (top!=null && *){
      //pop
      top=top.next ;
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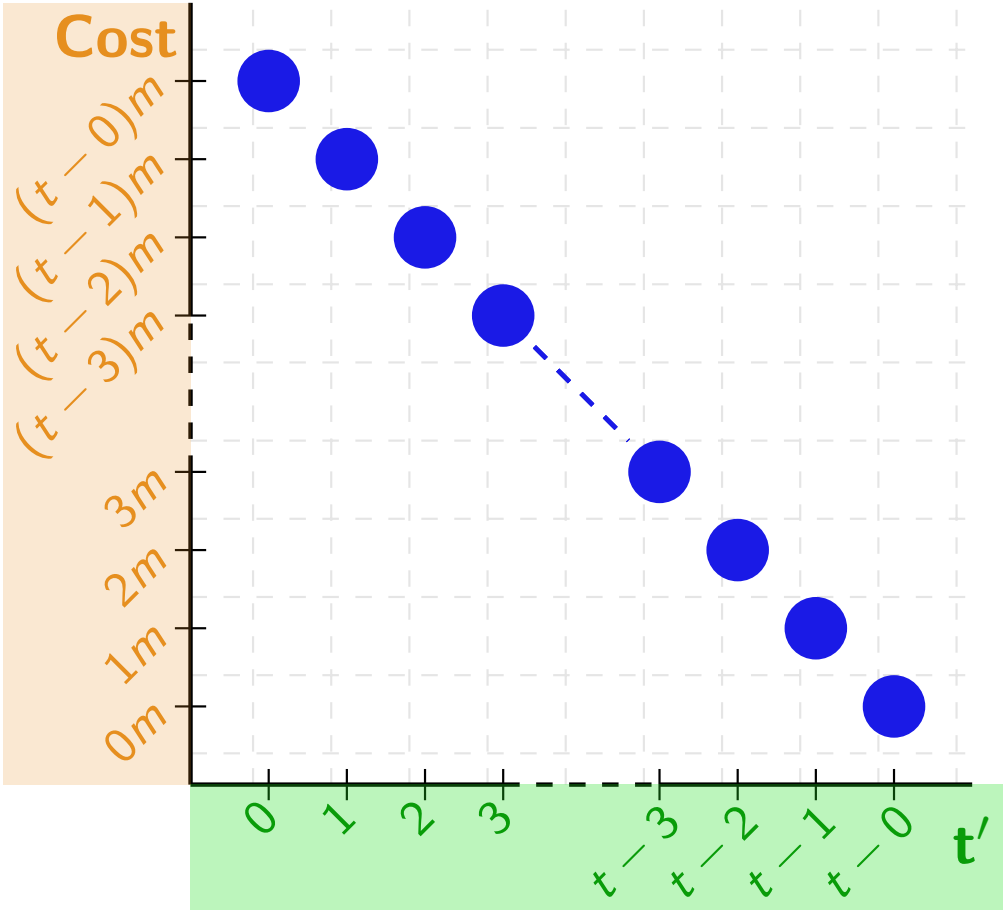
}}
```

The output-cost correlation

$$main(t,m) = 0 \quad \{m = 0, t \geq 0\}$$

$$main(t,m) = \begin{matrix} rpop(t,m) + 1 + \\ main(t'+1, m-1) \end{matrix} \quad \left\{ \begin{matrix} m \geq 1, t \geq 0 \\ t \geq t' \geq 0 \end{matrix} \right\}$$



The cost of *rpop* depends on the value of its output

- $t \geq t' \geq 0$ is a precise postcondition
- $0 \leq c \leq t * m$ is a precise cost relation

```
//@requires m >= 1
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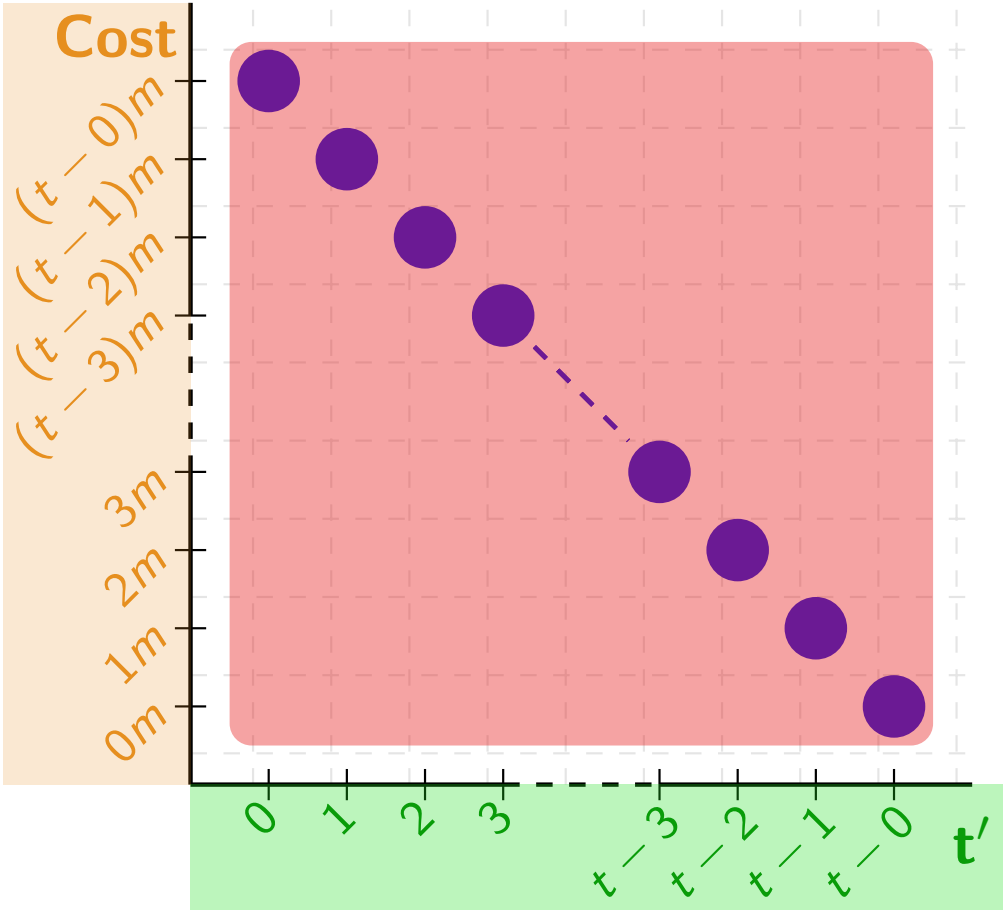
The output-cost correlation

$$main(t,m) = 0 \quad \{m = 0, t \geq 0\}$$

$$main(t,m) = \begin{matrix} rpop(t,m) + 1 + \\ main(t'+1, m-1) \end{matrix} \quad \left\{ \begin{matrix} m \geq 1, t \geq 0 \\ t \geq t' \geq 0 \end{matrix} \right\}$$



The cost of *rpop* depends on the value of its output

- $t \geq t' \geq 0$ is a precise postcondition
- $0 \leq c \leq t * m$ is a precise cost relation
- separating them introduces spurious evaluations

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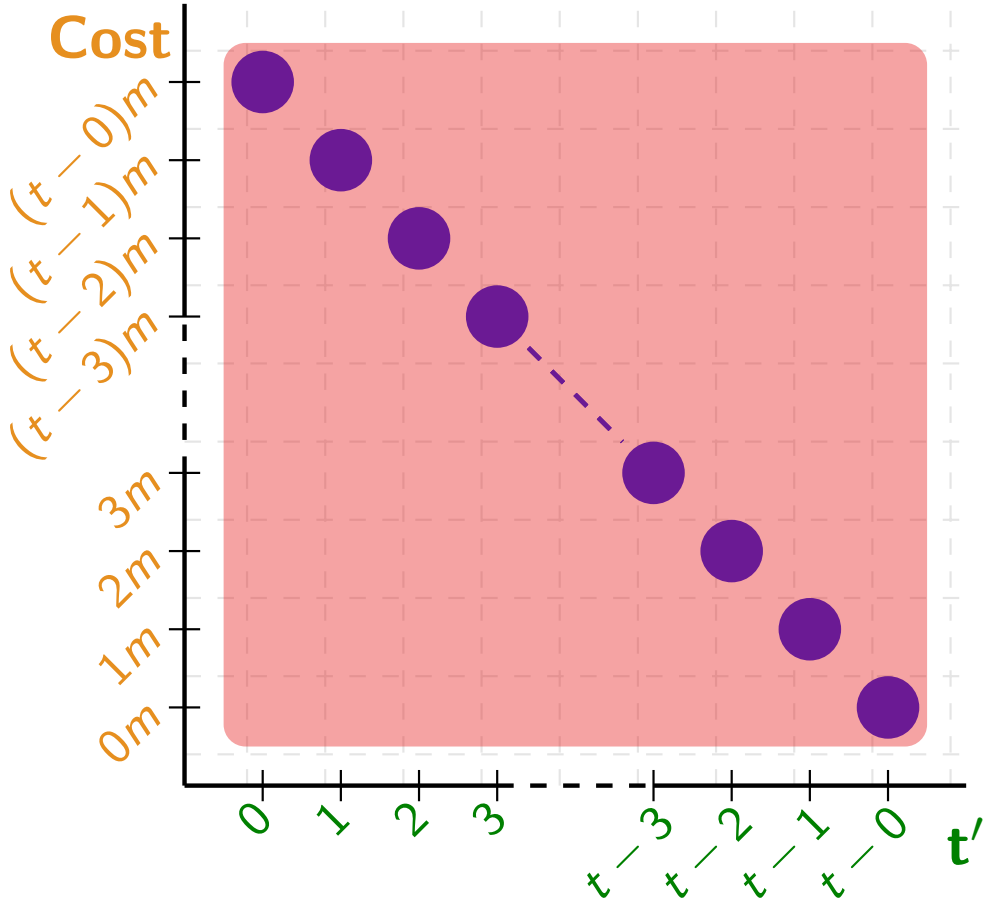
```

The output-cost correlation

The cost of *rpop* depends on the value of its output

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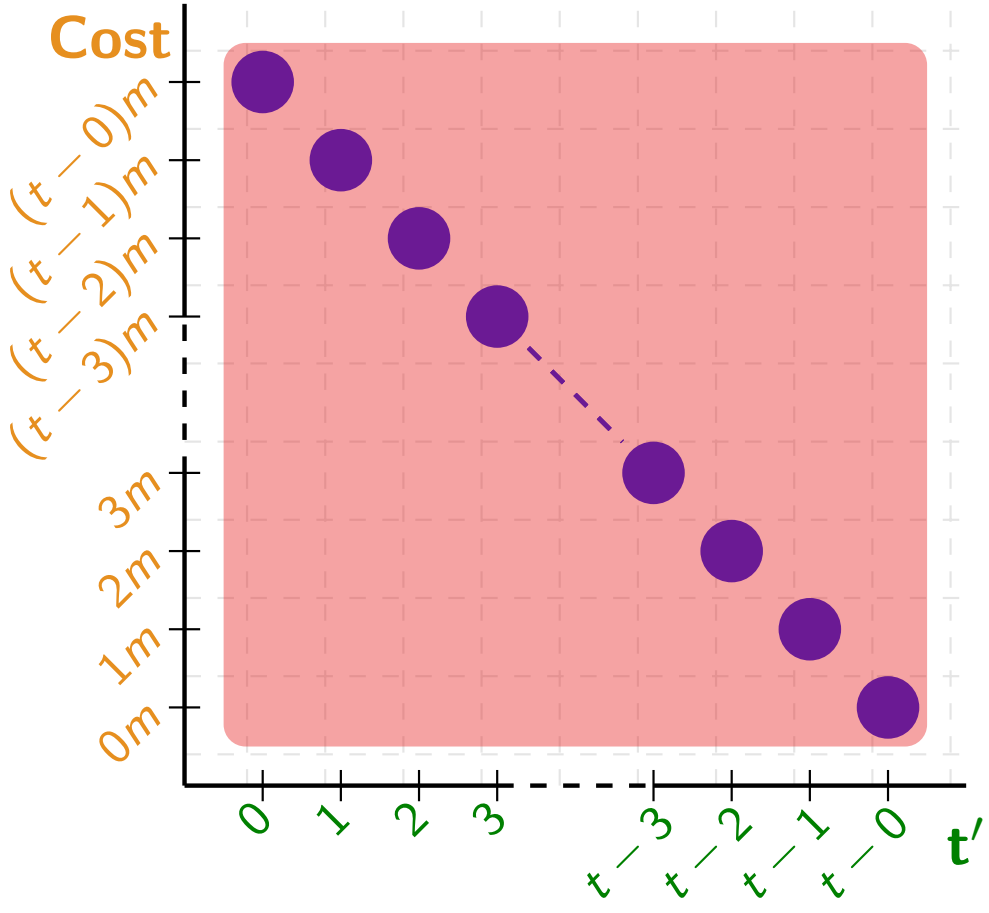
}}
```

The output-cost correlation

$$rpop^u(t, m) = m * t$$



The cost of *rpop* depends on the value of its output

- $c \leq m * t$ is best cost UB that takes *rpop* inputs

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//@requires m >= 1
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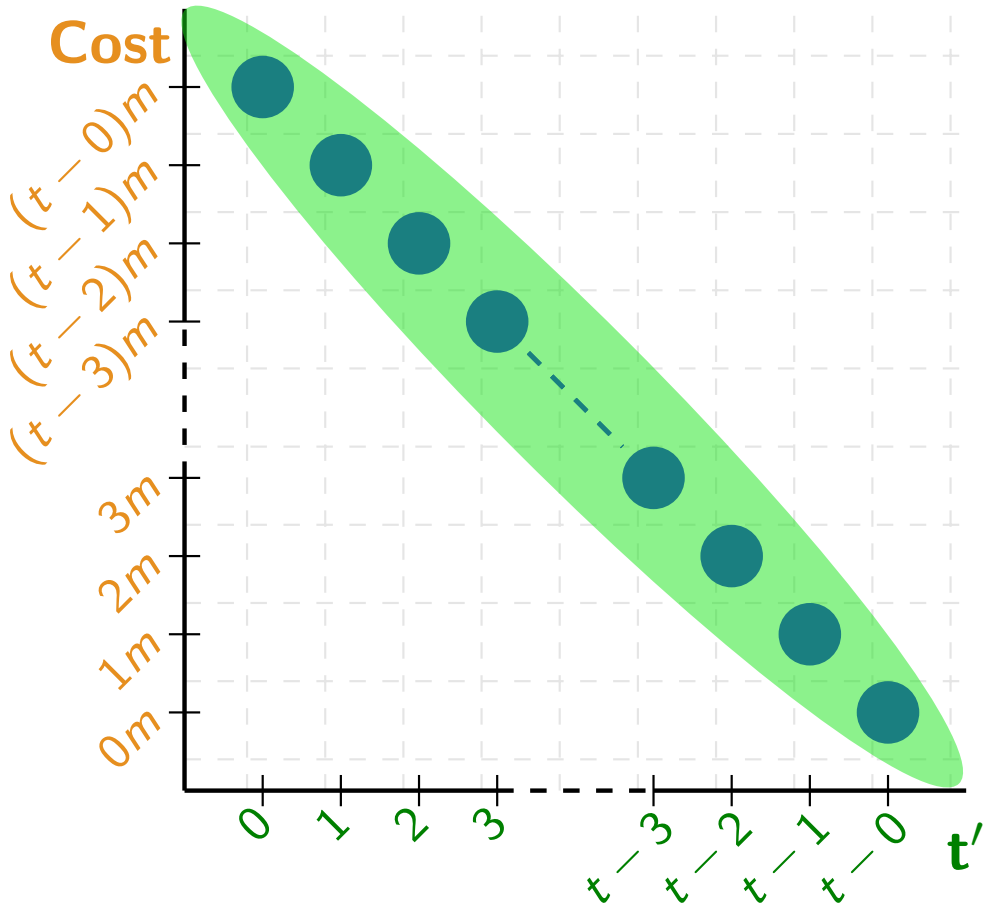
The output-cost correlation

$$rpop^u(t, m, t') = m(t - t')$$



The cost of *rpop* depends on the value of its output

- $c \leq m * t$ is best cost UB that takes *rpop* inputs
- $c = m * (t - t')$ is an exact cost that takes both *rpop* inputs **and outputs**

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//@requires m >= 1
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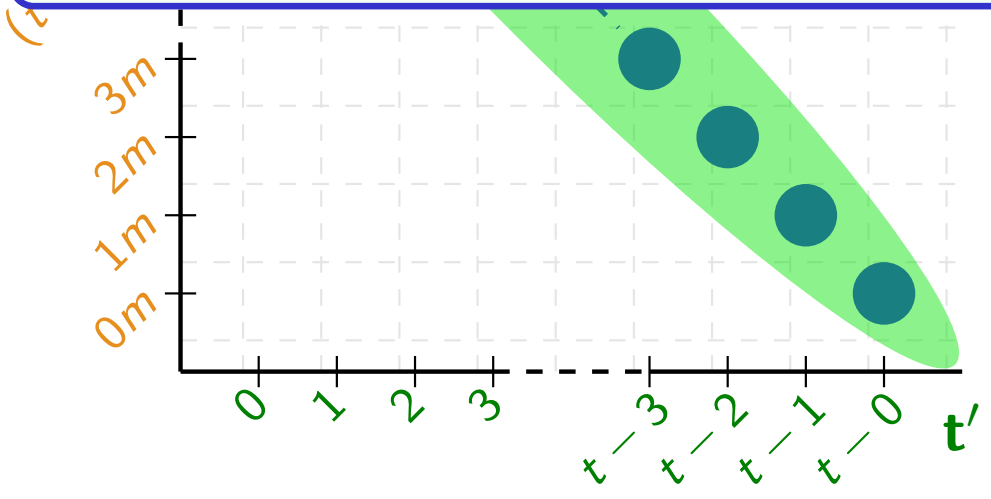
The output-cost correlation

$$rpop^u(t, m, t') = m(t - t')$$

Net Cost Upper Bound

Every *rpop* execution that starts with s, m and terminates at t' consumes at most $m * (t - t')$ resources



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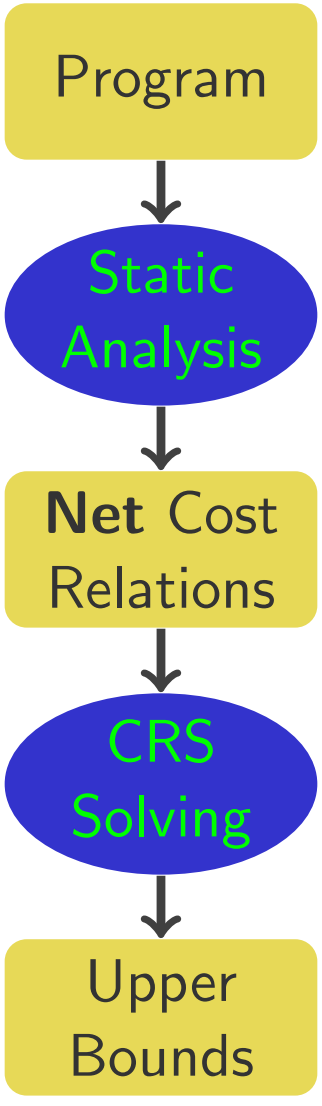
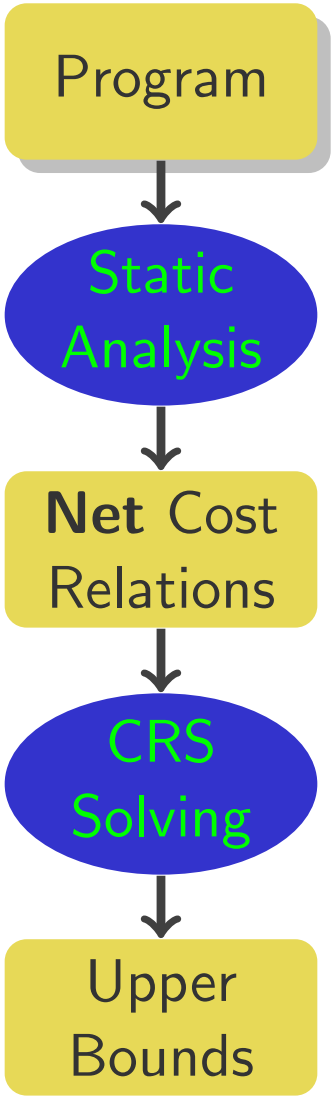
```

Classical Approach to Cost Analysis: repaired example



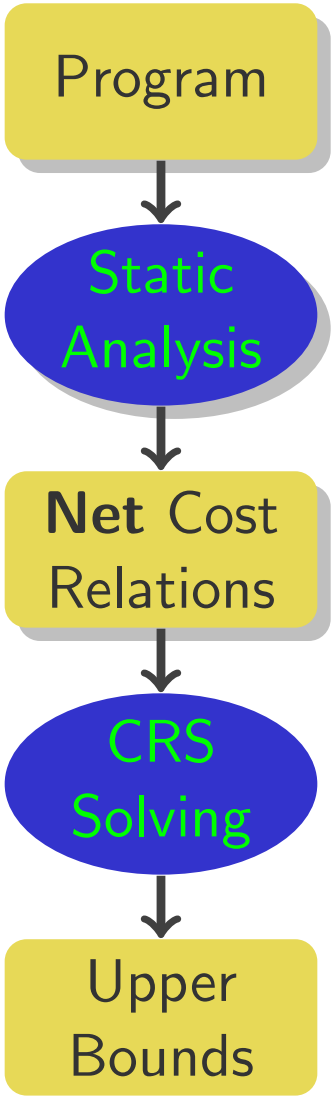
Classical Approach to Cost Analysis: repaired example




```
//@requires m >= 1
void rpop(int m){
  if (top != null && *){
    //pop
    top=top.next ;
    //@acquire(m)
}}
```

```
//@requires m >= 0
void main(int m) {
  while( m > 0 ) {
    rpop(m)  ;
    // push(*);
    top = new
     Node(*,top);
    //@acquire(1)
    m = m-1      ;
}}}
```

Classical Approach to Cost Analysis: repaired example

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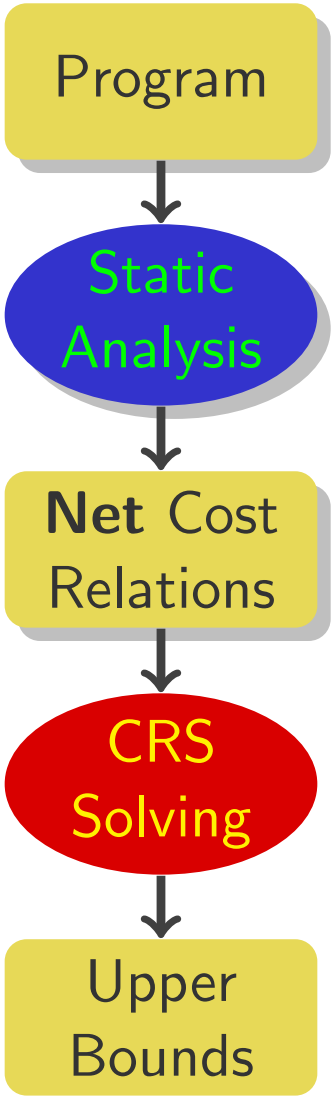
    top = new
     Node(*,top);
    //@acquire(1)
    m = m-1     ;
}}}
```

$$
\begin{aligned}
main(t, m, t') &= 0 && \{m = 0, t \geq 0, t' = t\} \\
main(t, m, t_1) &= rpop(t, m, t') + 1 + main(t' + 1, m - 1, t_1) && \left\{\begin{matrix} m \geq 1, t \geq 0 \\ t \geq t' \geq 0 \end{matrix}\right\} \\
rpop(t, m, t') &= 0 && \{m \geq 1, t \geq 0, t' = t\} \\
rpop(t, m, t') &= m + rpop(t - 1, m, t') && \{m \geq 1, t \geq 1\}
\end{aligned}
$$

Classical Approach to Cost Analysis: repaired example

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     Node(*,top);
```

Net Cost Upper Bound

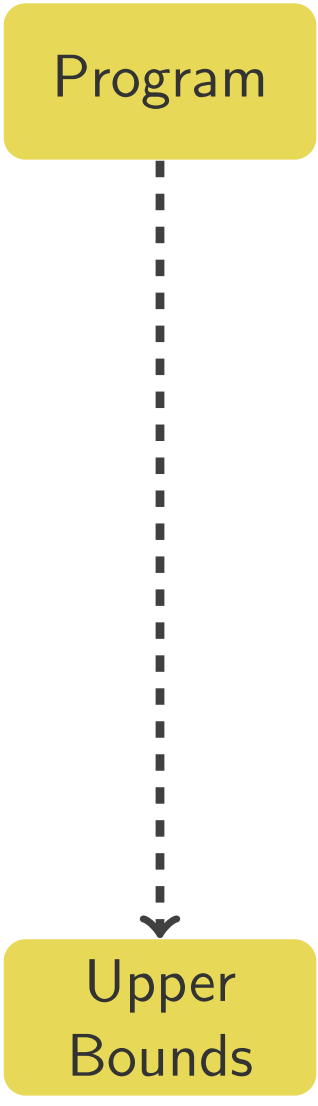
Existing CR Solvers do not obtain precise bounds for net cost equations

$$main(t, m, t_1) = \begin{matrix} rpop(t, m, t') + 1 + \\ main(t' + 1, m - 1, t_1) \end{matrix} \quad \begin{Bmatrix} m \geq 1, t \geq 0 \\ t \geq t' \geq 0 \end{Bmatrix}$$

$$rpop(t, m, t') = 0 \quad \{m \geq 1, t \geq 0, t' = t\}$$

$$rpop(t, m, t') = m + rpop(t - 1, m, t') \quad \{m \geq 1, t \geq 1\}$$

New Approach to Cost Analysis

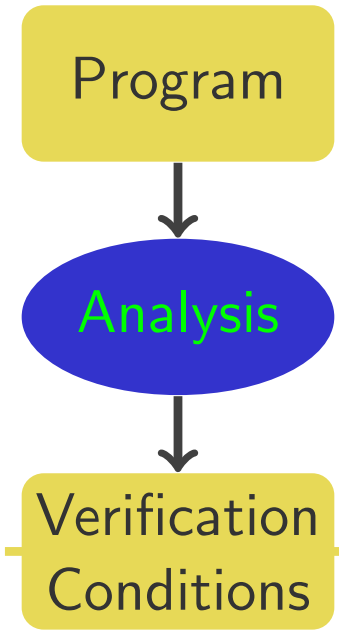


New Approach to Cost Analysis

Our approach uses the **Verification Conditions** to split cost analysis in two independent phases

New Approach to Cost Analysis

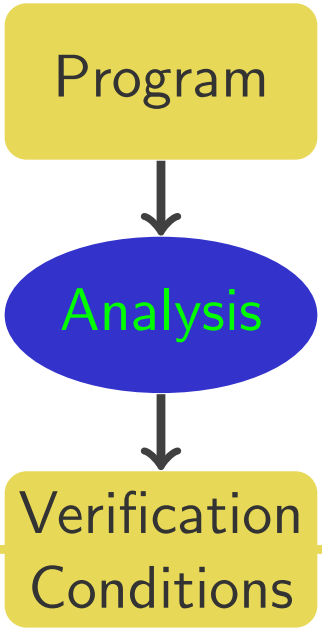
New Approach to Cost Analysis



An **analysis** abstracts the program semantics into a system of inductive **Verification Conditions**

Upper Bounds

New Approach to Cost Analysis



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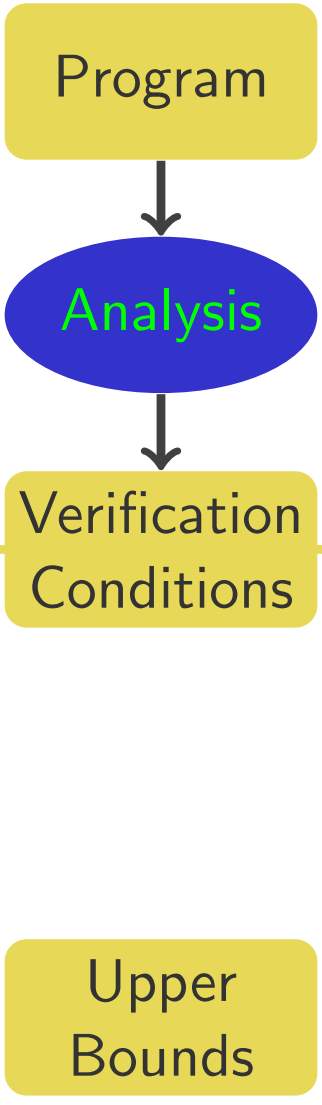
Verification Conditions describe properties a function must hold to be a valid net cost bound

Upper Bounds

New Approach to Cost Analysis

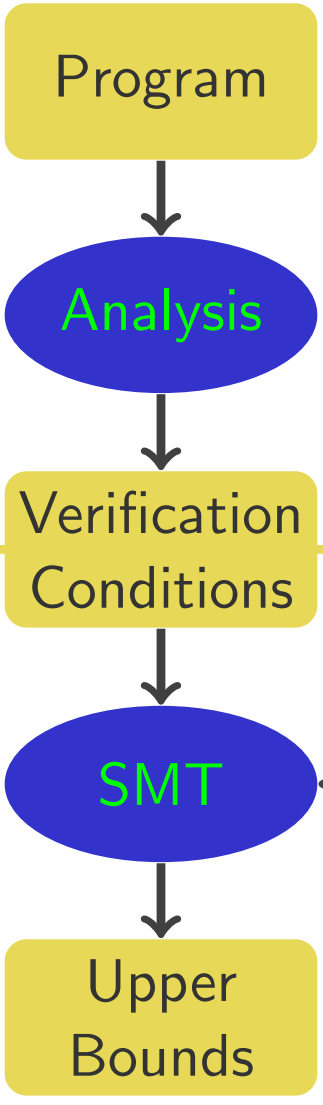


An **analysis** abstracts the program semantics into a system of inductive **Verification Conditions**

Verification Conditions describe properties a function must hold to be a valid net cost bound

Verification Conditions are formulas in First Order logic, universally quantified over the program variables

New Approach to Cost Analysis



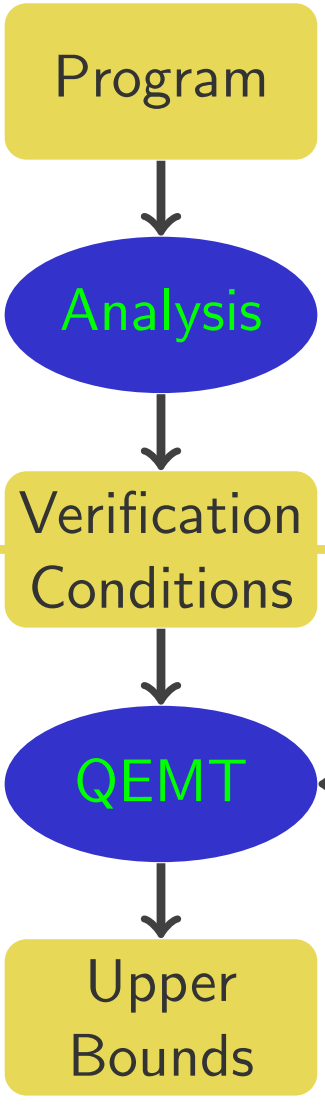
An **analysis** abstracts the program semantics into a system of inductive **Verification Conditions**

Verification Conditions describe properties a function must hold to be a valid net cost bound

Verification Conditions are formulas in First Order logic, universally quantified over the program variables

We can **Verify** a set of candidate Upper Bounds by checking the **Validity Modulo Theory** of the VCs

New Approach to Cost Analysis



An **analysis** abstracts the program semantics into a system of inductive **Verification Conditions**

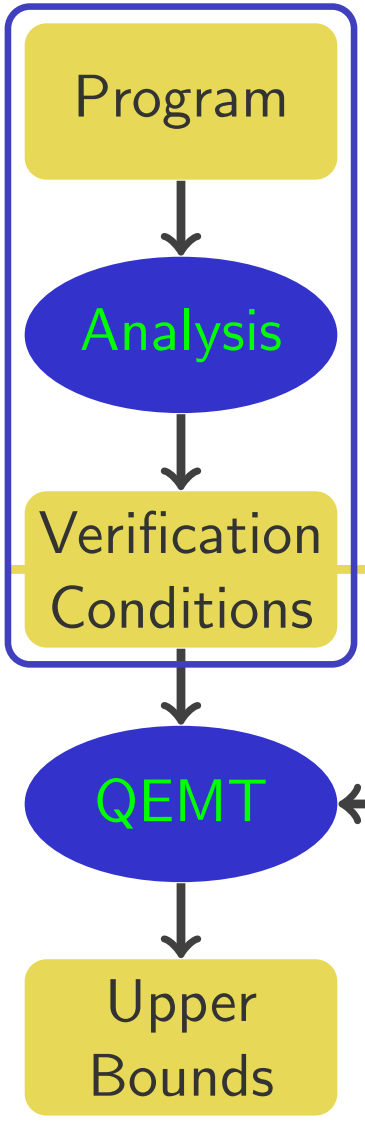
Verification Conditions describe properties a function must hold to be a valid net cost bound

Verification Conditions are formulas in First Order logic, universally quantified over the program variables

Template Upper Bound

With some template UBs, we can **synthesise** concrete bounds using **Quantifier Elimination** modulo Theory

New Approach to Cost Analysis



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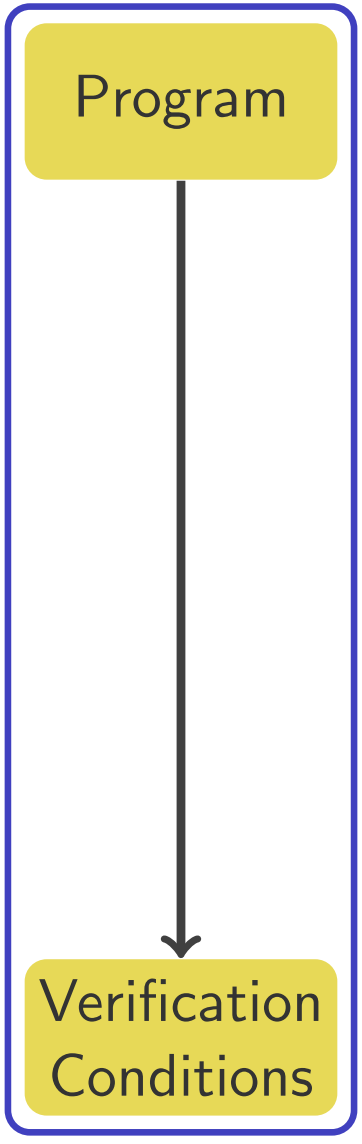
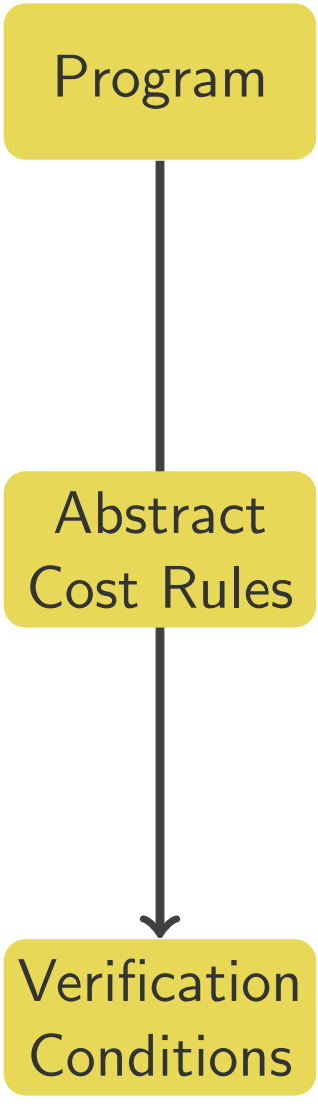
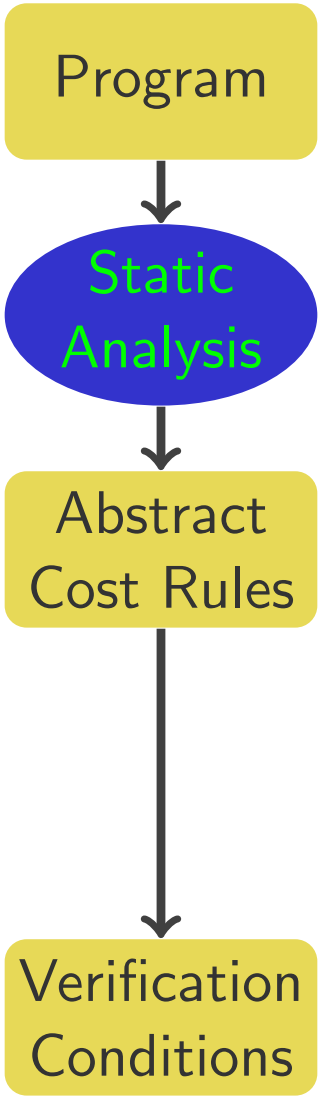


New Approach to Cost Analysis



We use **Abstract Cost Rules** as an intermediate abstraction of the program

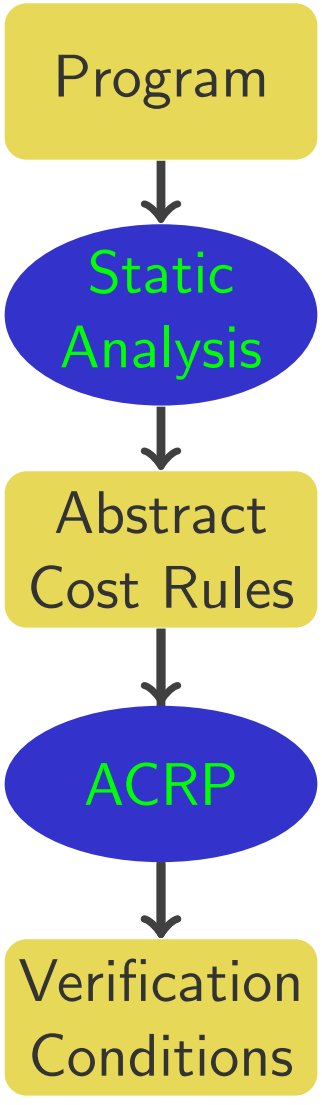
New Approach to Cost Analysis



The existing static analysis generates, **Abstract Cost Rules** as an intermediate representation

We use **Abstract Cost Rules** as an intermediate abstraction of the program

New Approach to Cost Analysis

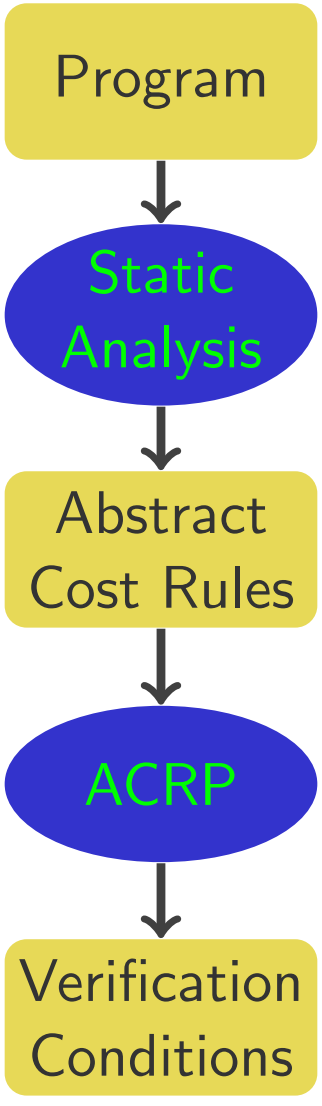


The existing static analysis generates, **Abstract Cost Rules** as an intermediate representation

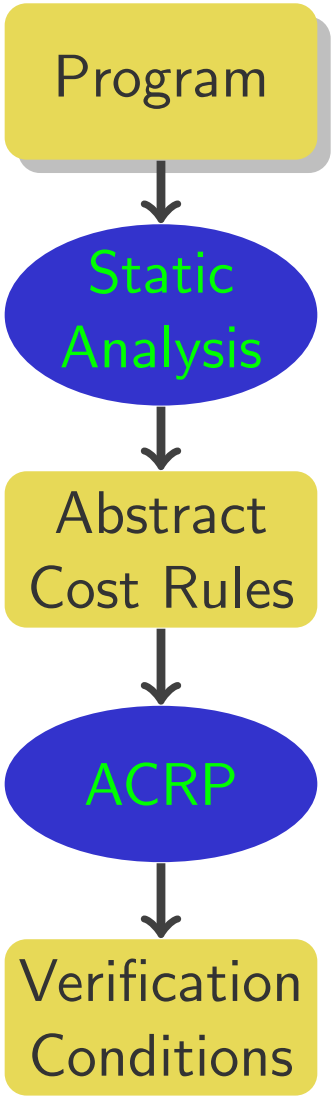
We use **Abstract Cost Rules** as an intermediate abstraction of the program

An Abstract Cost Rules Processor **(ACRP)** transforms each ACR into a VC

New Approach to Cost Analysis



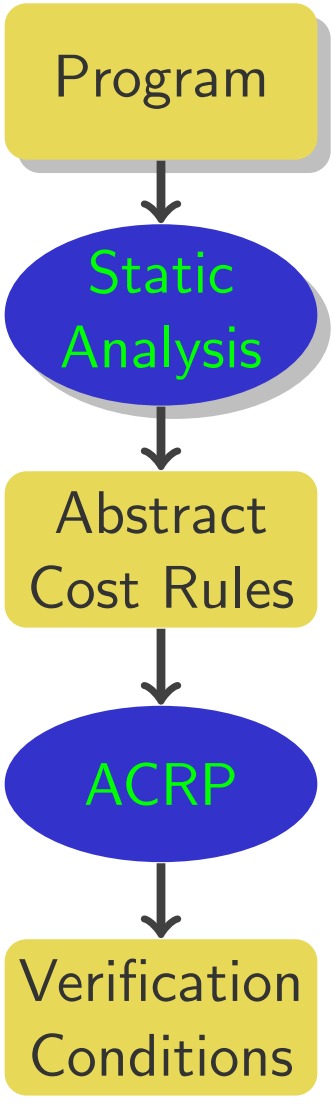
New Approach to Cost Analysis




```
//@requires m >= 0
void main(int m) {
  while( m > 0 ) {
    rpop(m)  ;
    // push(*);
    top = new
     Node(*,top);
    //@acquire(1)
    m = m-1      ;
}}}
```

```
//@requires m >= 1
void rpop(int m){
  while(top!=null && *){
    //pop
    top=top.next ;
    //@acquire(m)
}}
```

New Approach to Cost Analysis

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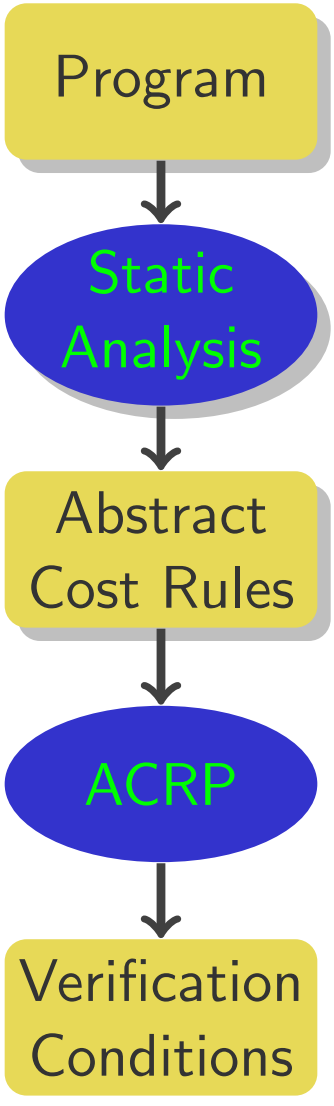


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```

$$main([t, 0], [t]) \leftarrow$$
$$\quad t \geq 0.$$
$$main([t, m], [t_1]) \leftarrow$$
$$\quad t \geq 0,$$
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$$\quad rpop([t, m], [t']),$$
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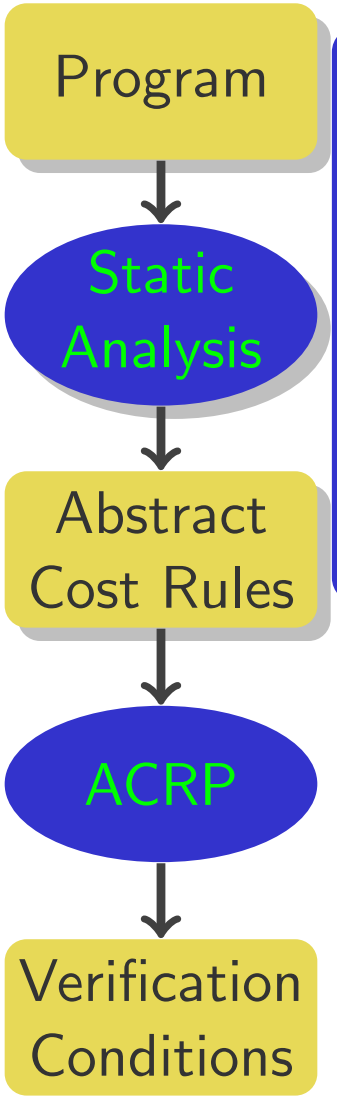
New Approach to Cost Analysis



Abstract Cost Rules

An ACR has the form $p(\bar{x}|\bar{y}) \longleftarrow b_1, \ldots b_n$.

There are three kinds of abstract codes b_i

Constraint	φ	Operations & guards
Cost	$\mathtt{acq}(e)$	Resource consumption
Calls	$q(\bar{x}|\bar{z})$	Control Flow

$$
\begin{array}{l}
main([t,0],[t]) \leftarrow \\
\quad t \geq 0. \\
main([t,m],[t_1]) \leftarrow \\
\quad t \geq 0, \\
\quad m \geq 1, \\
\quad rpop([t,m],[t']), \\
\quad \mathtt{acq}(1), \\
\quad main([t'+1, m-1],[t_1]).
\end{array}
$$

$$
\begin{array}{l}
rpop([t,m],[t]) \leftarrow \\
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\quad m \geq 1. \\
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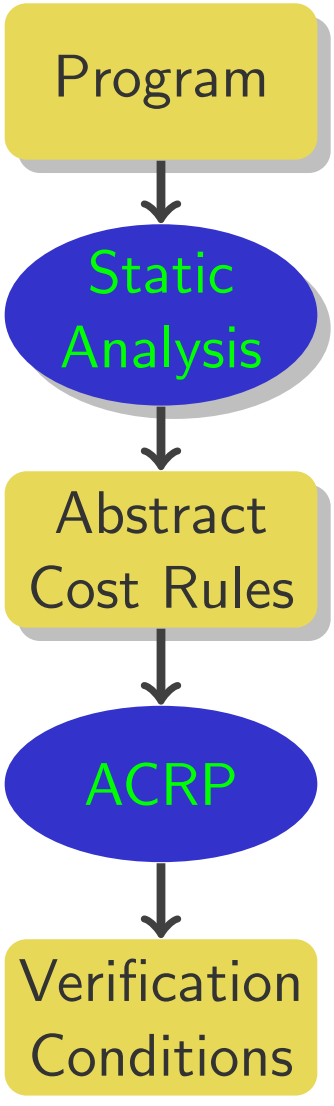
\quad t \geq 1, \\
\quad m \geq 1, \\
\quad \mathtt{acq}(m), \\
\quad rpop([t-1,m],[t']).
\end{array}
$$

New Approach to Cost Analysis




```
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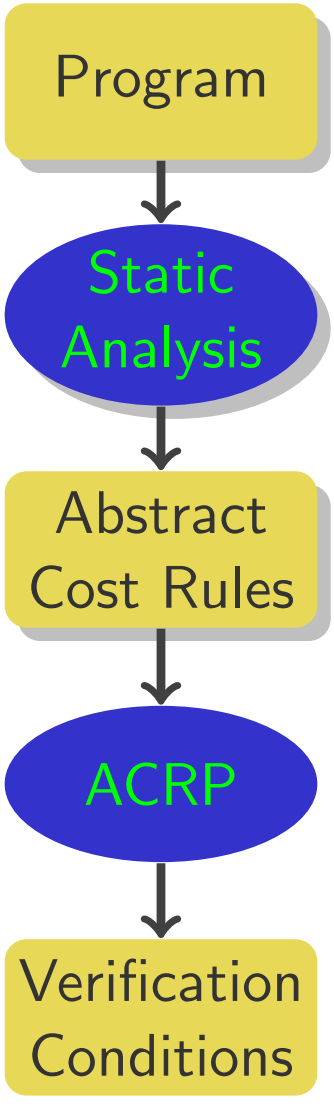


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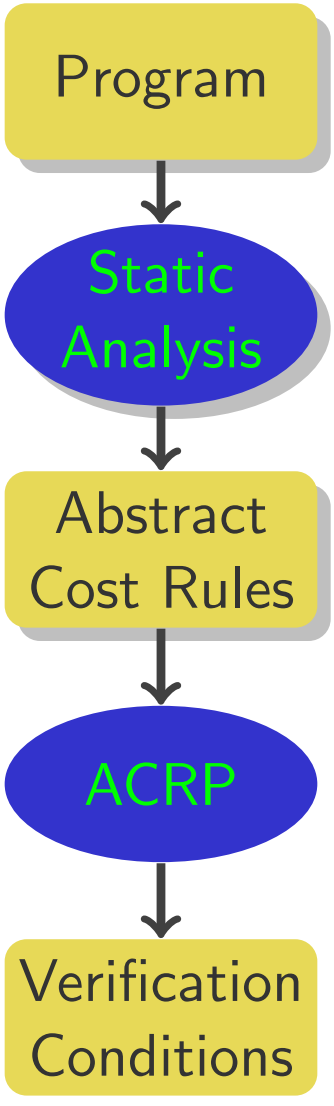


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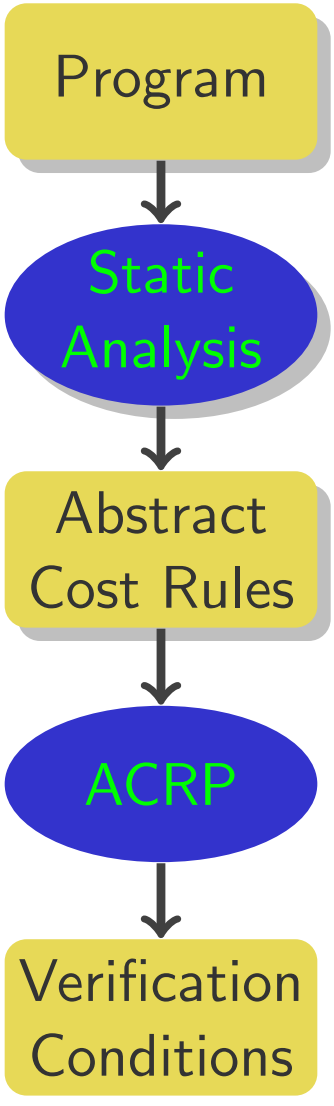


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$$

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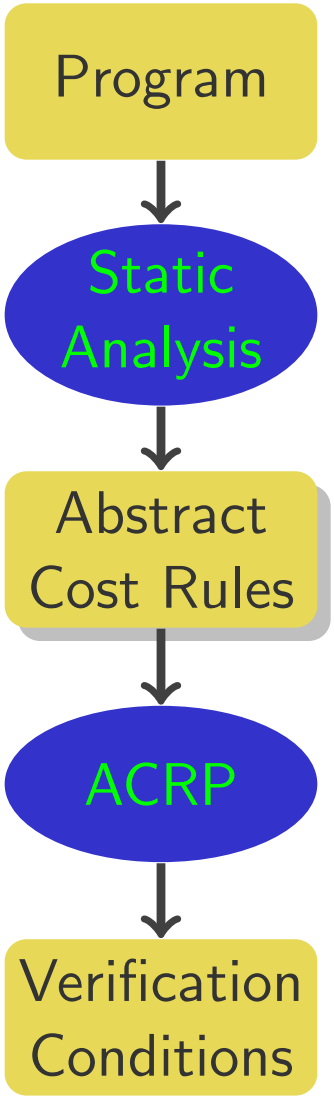

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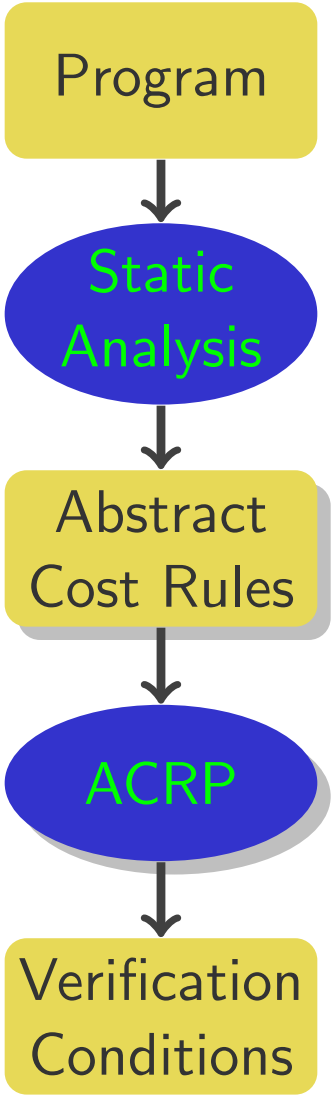

$$\frac{\forall * . \wedge_{i=1}^{n} \varphi_i \rightarrow \tilde{f}_p(\bar{x}, \bar{y}) \geq \sum_{i=1}^{n} \tilde{b}_i}{p(\bar{x}|\bar{y}) \longleftarrow b_1, \ldots b_n}$$

$main([t, 0], [t]) \leftarrow$
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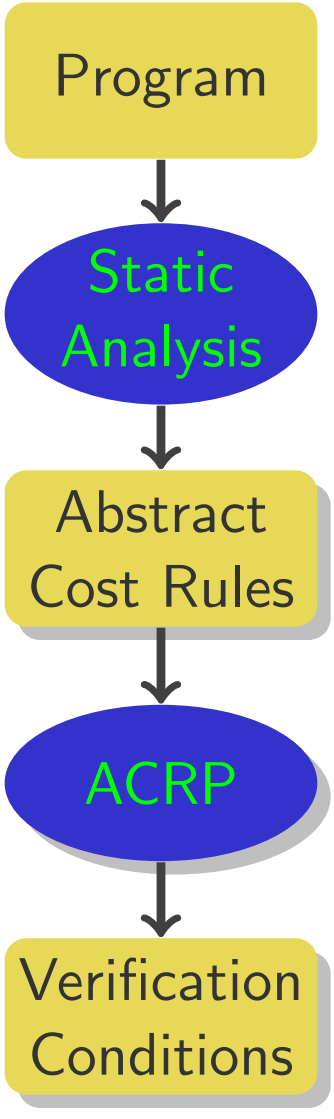

$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$
$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$
$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

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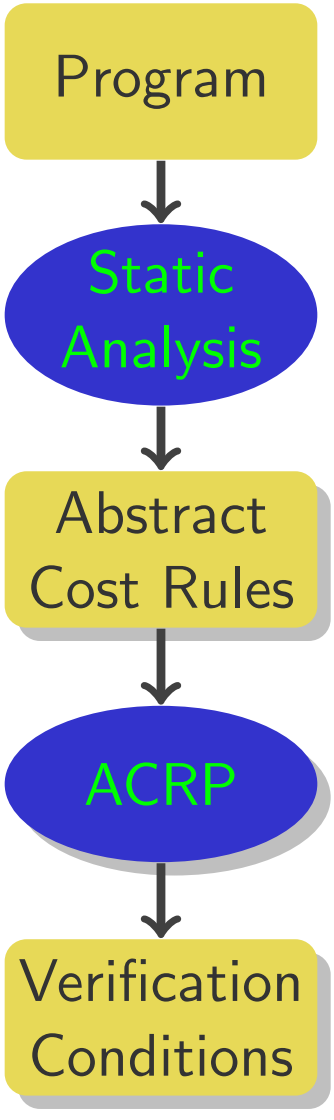
New Approach to Cost Analysis



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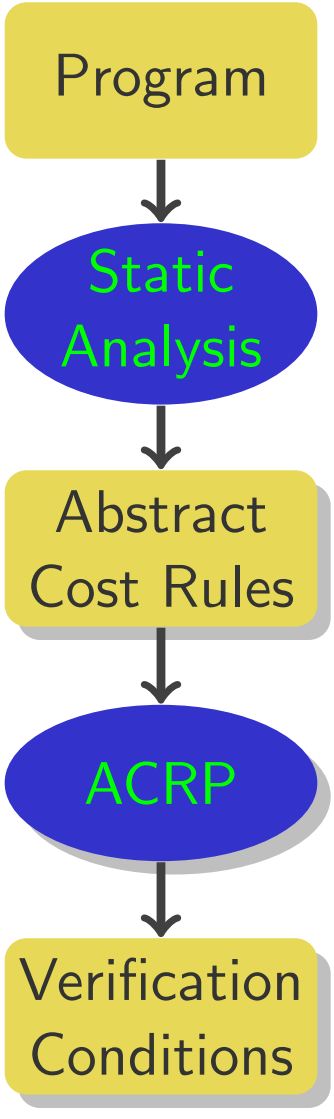
New Approach to Cost Analysis



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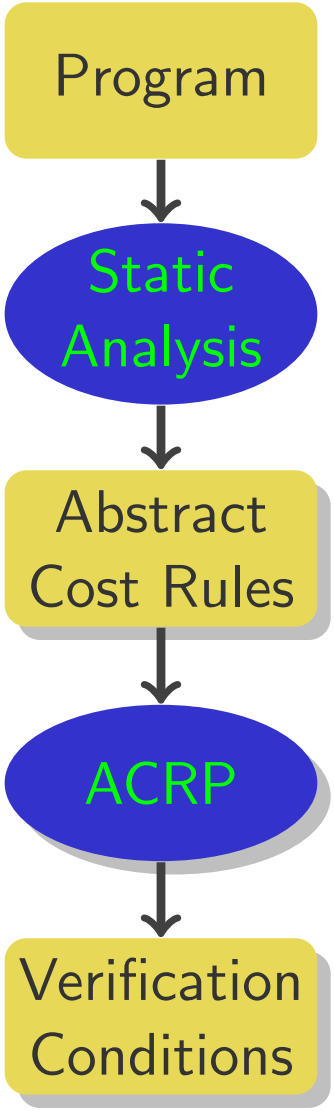
New Approach to Cost Analysis



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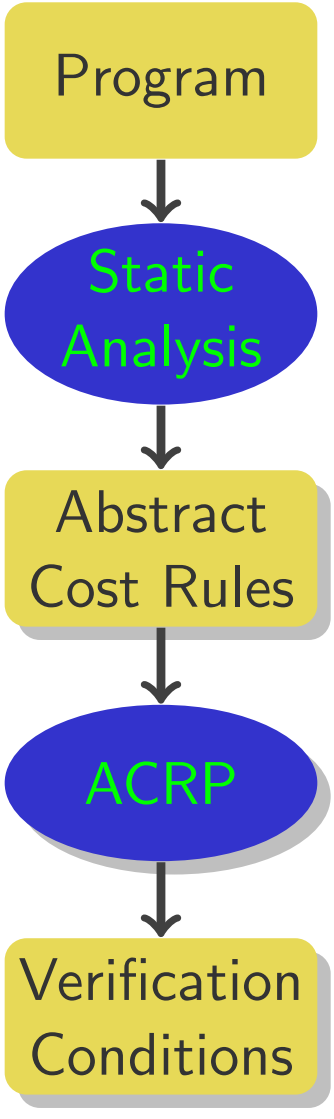
New Approach to Cost Analysis



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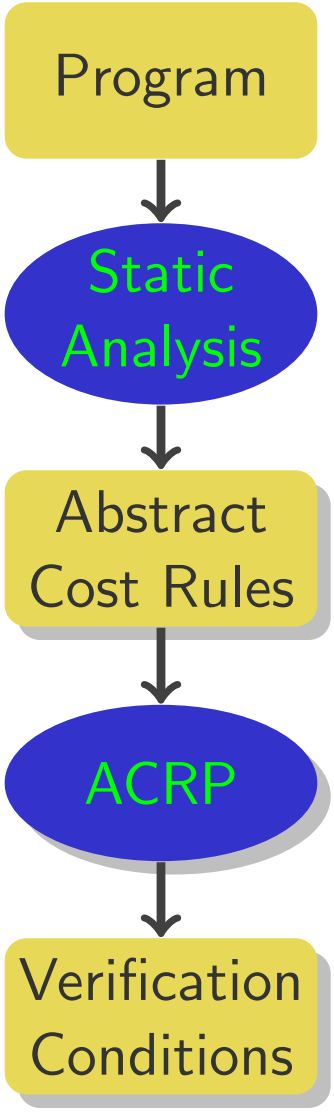
New Approach to Cost Analysis



$$\begin{array}{ll}
\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow & \tilde{f}_{main}(t, m, t) \geq 0 \\
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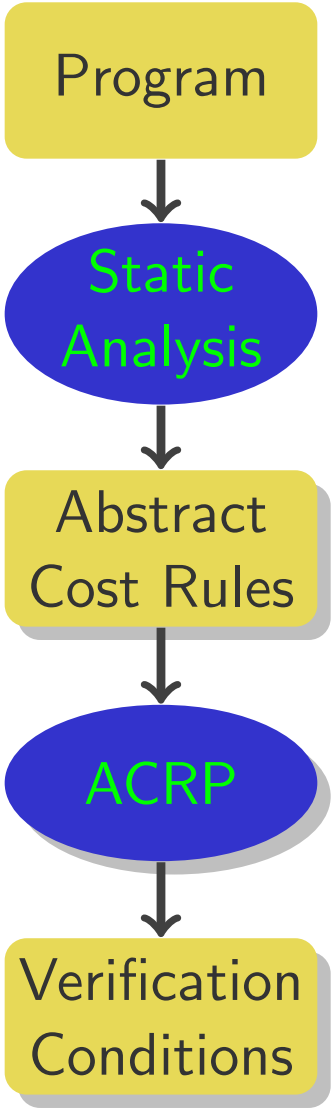
New Approach to Cost Analysis



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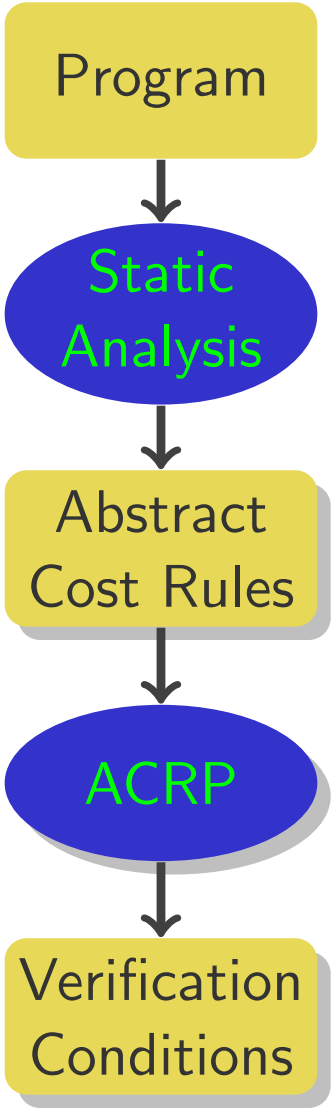
New Approach to Cost Analysis



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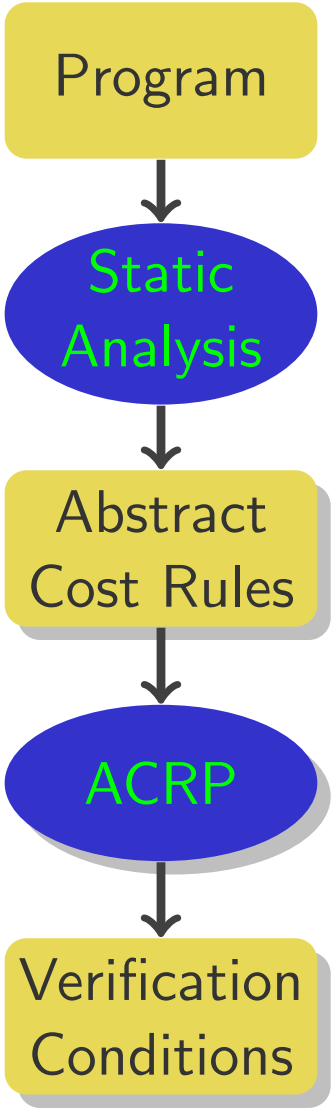
New Approach to Cost Analysis



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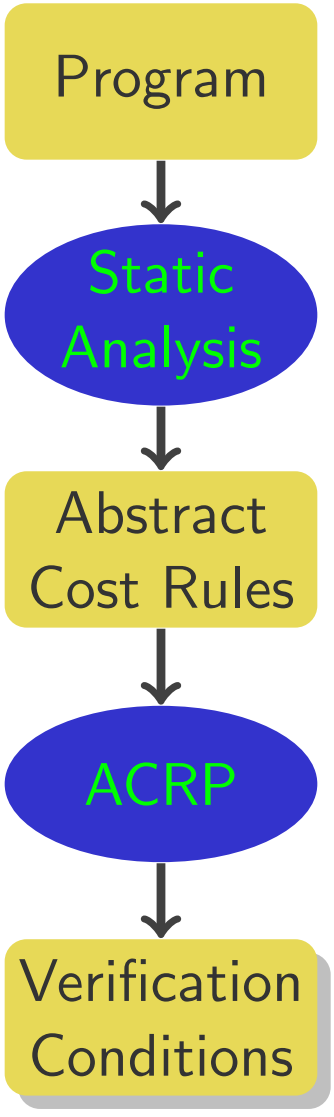
New Approach to Cost Analysis



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New Approach to Cost Analysis

Verification Conditions

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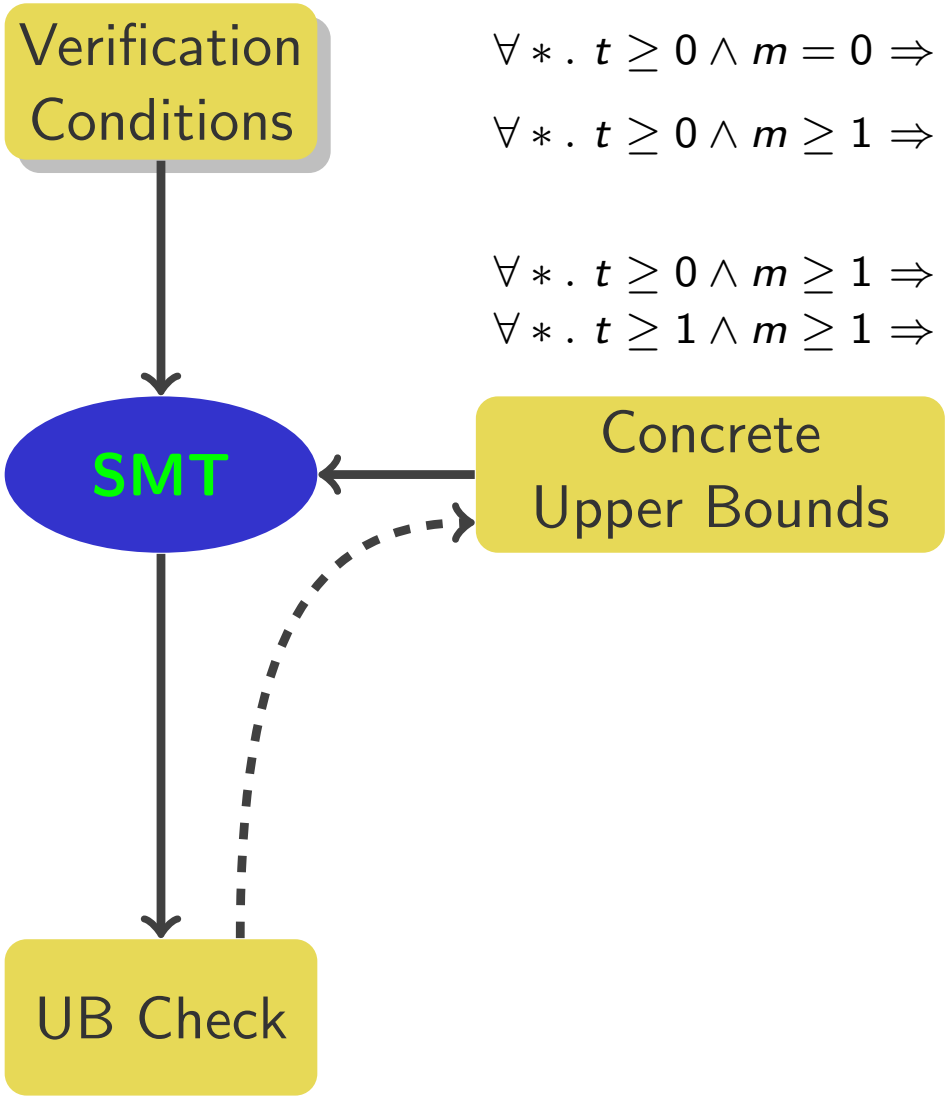
New Approach to Cost Analysis



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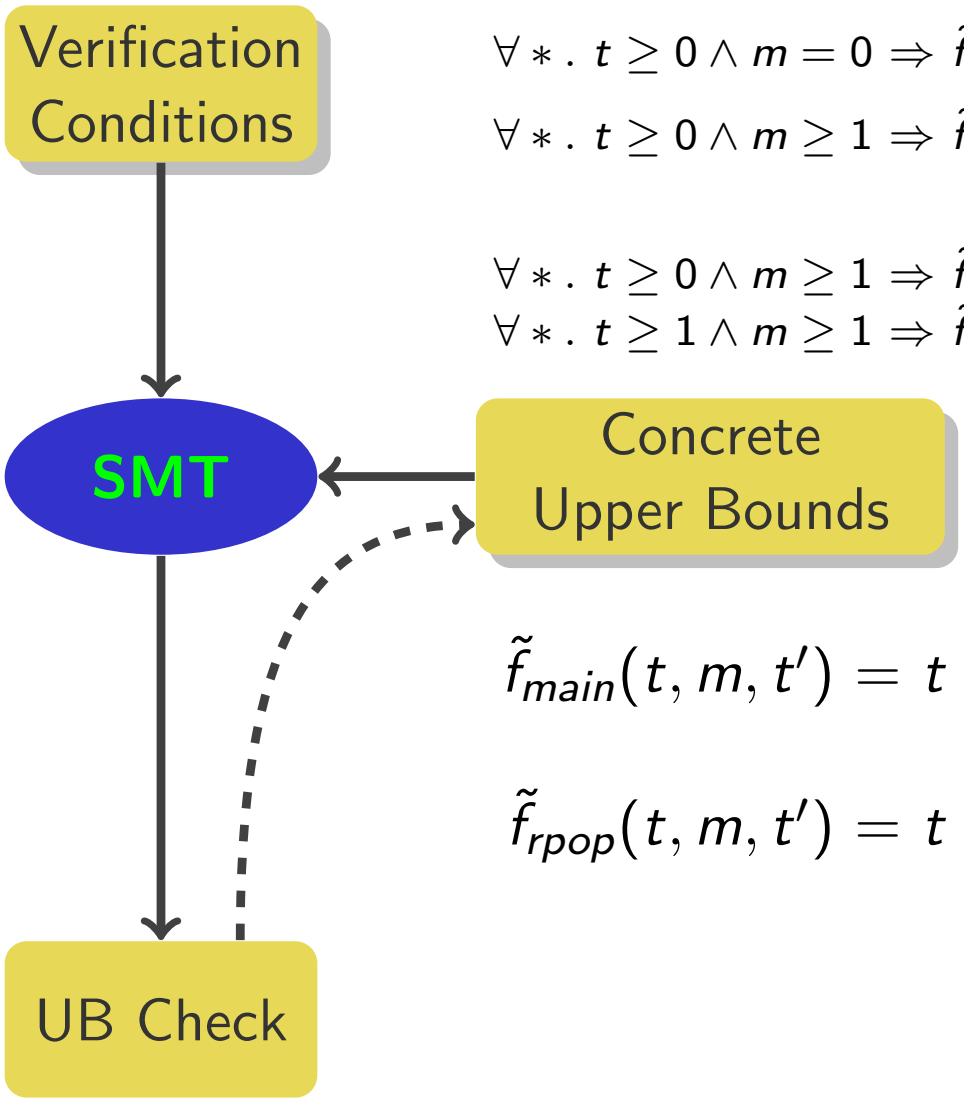
New Approach to Cost Analysis



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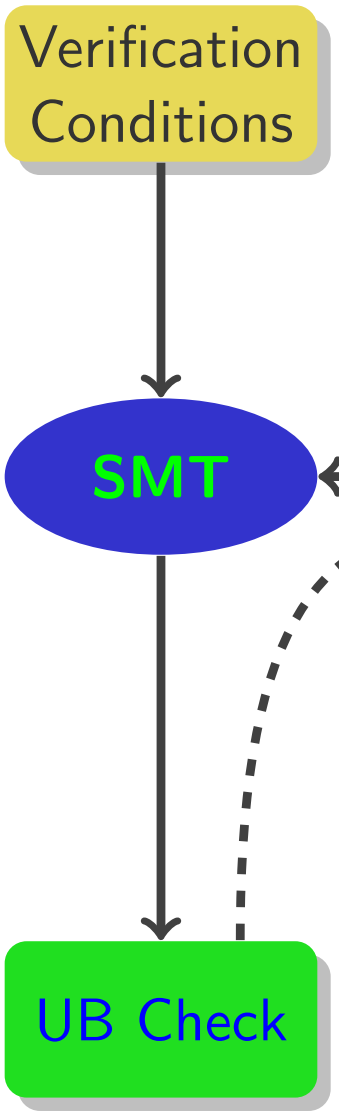
1. We try these Input & Output Upper Bounds

$$\tilde{f}_{main}(t, m, t') = t * m + \frac{m^2 + m}{2}$$

$$\tilde{f}_{rpop}(t, m, t') = t * m - t' * m$$

New Approach to Cost Analysis



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Concrete Upper Bounds

1. We try these Input & Output Upper Bounds

$$\tilde{f}_{main}(t, m, t') = t * m + \frac{m^2 + m}{2}$$

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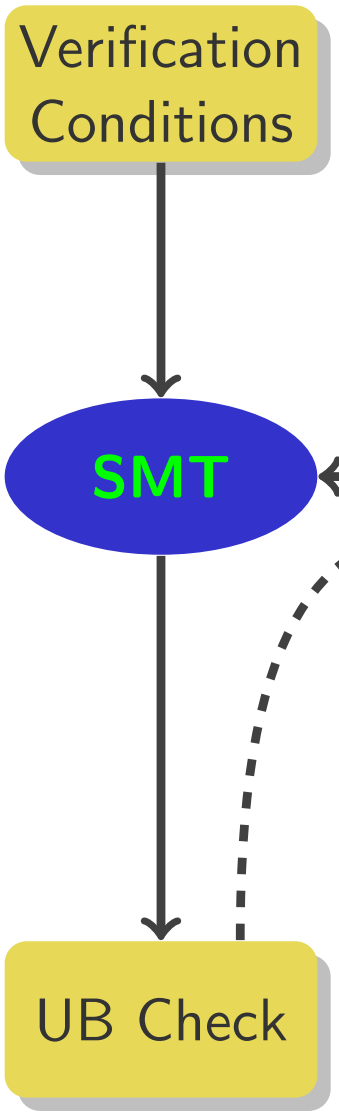
Valid

New Approach to Cost Analysis



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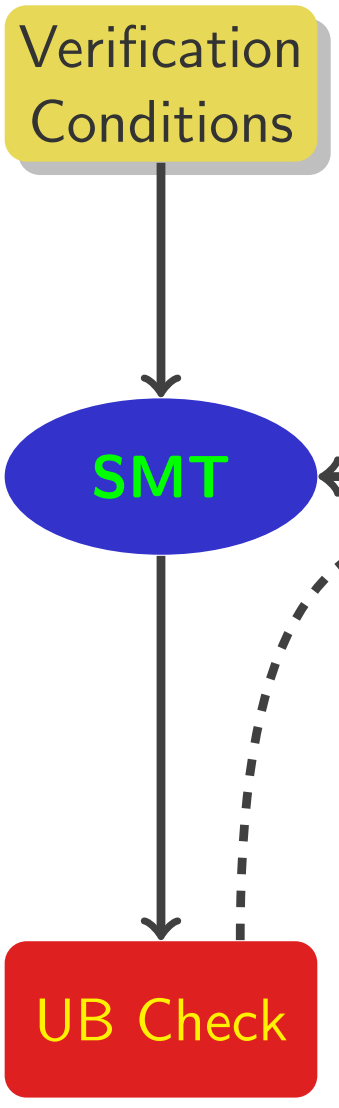
Concrete Upper Bounds

2. We try the desired Input-only Upper Bounds

$$\tilde{f}_{main}(t, m, t') = t * m + \frac{m^2 + m}{2}$$

$$\tilde{f}_{rpop}(t, m, t') = t * m$$

New Approach to Cost Analysis



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Concrete Upper Bounds

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$$\tilde{f}_{main}(t, m, t') = t * m + \frac{m^2 + m}{2}$$

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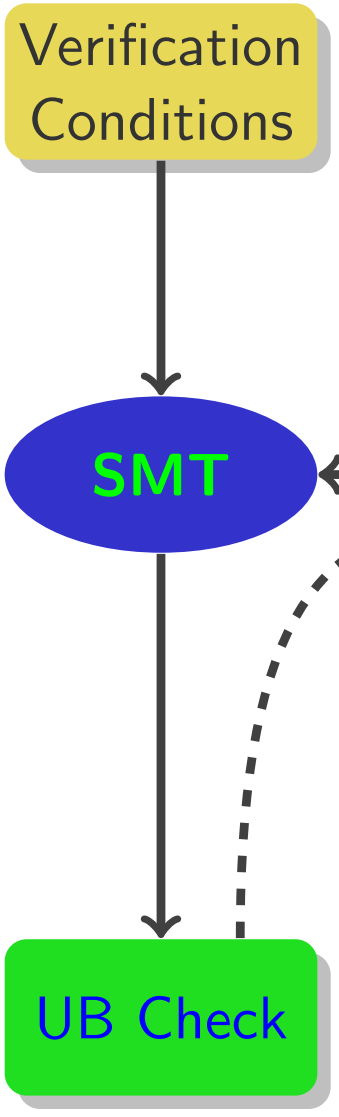
Invalid

New Approach to Cost Analysis



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Concrete Upper Bounds

2. We try the desired Input-only Upper Bounds

$$\tilde{f}_{main}(t, m, t') = t * m + \frac{m^2 + m}{2}$$

$$\tilde{f}_{rpop}(t, m, t') = t * m$$

Valid

New Approach to Cost Analysis

Verification Conditions

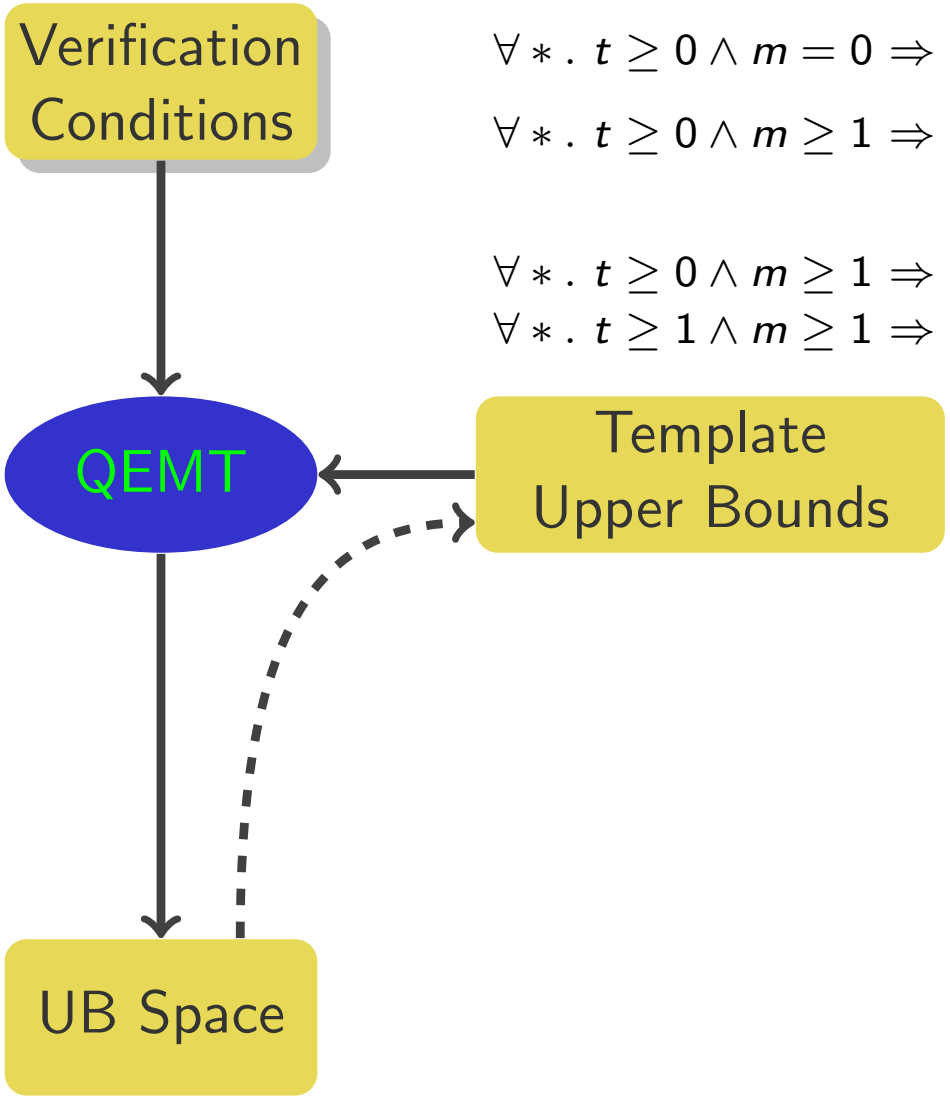
$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

New Approach to Cost Analysis



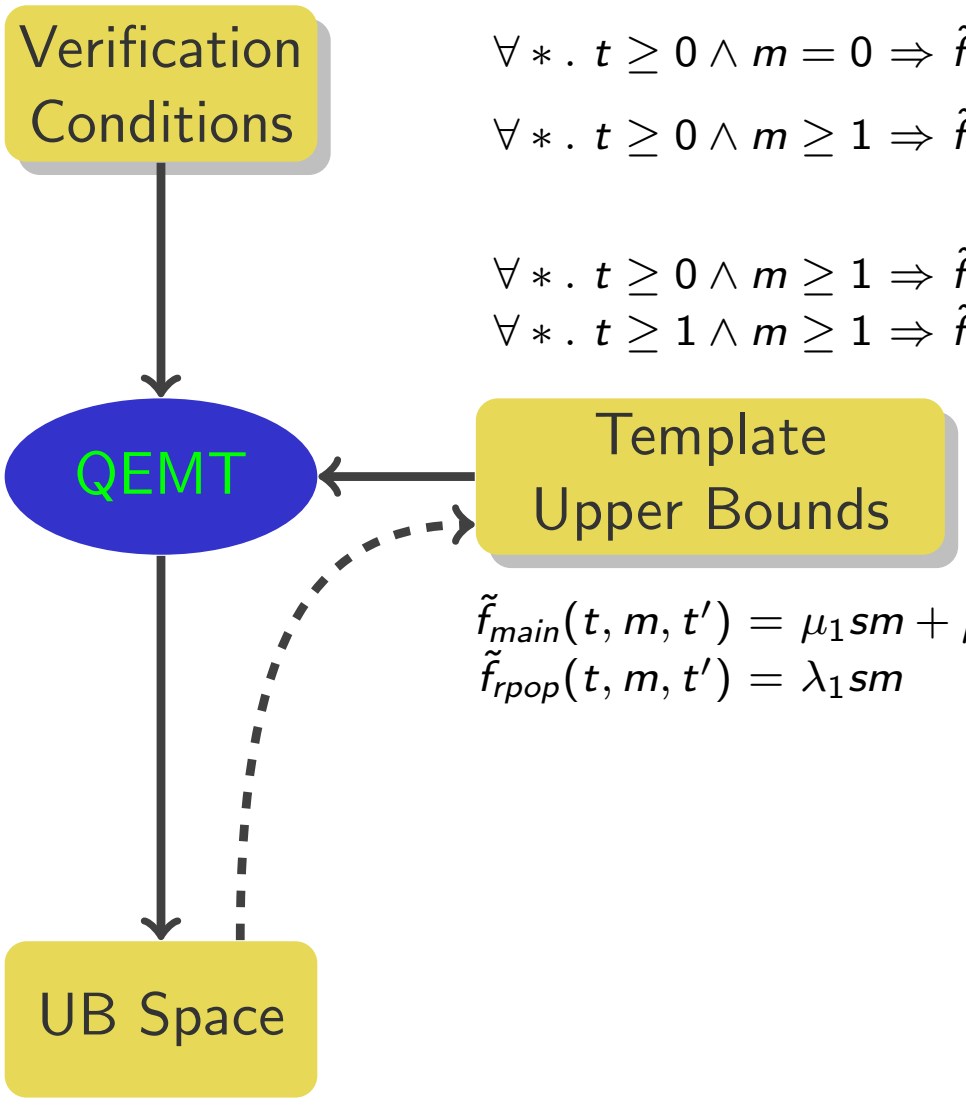
$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

New Approach to Cost Analysis



$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$

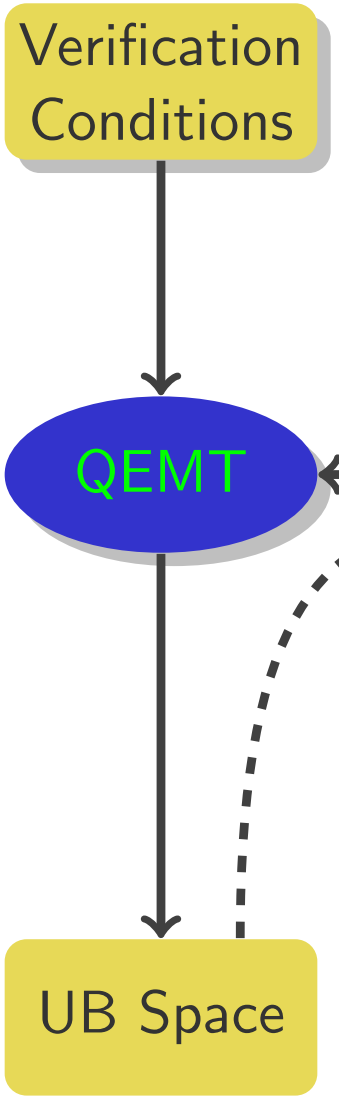
$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

1. We use these templates for Input & Output UBs

$$\tilde{f}_{main}(t, m, t') = \mu_1 sm + \mu_2 m^2 + \mu_3 mt' + \mu_4 s + \mu_5 m + \mu_6 t' + \mu_0$$

$$\tilde{f}_{rpop}(t, m, t') = \lambda_1 sm + \lambda_2 mt' + \lambda_3 s + \lambda_4 m + \lambda_5 t' + \lambda_0$$

New Approach to Cost Analysis



$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

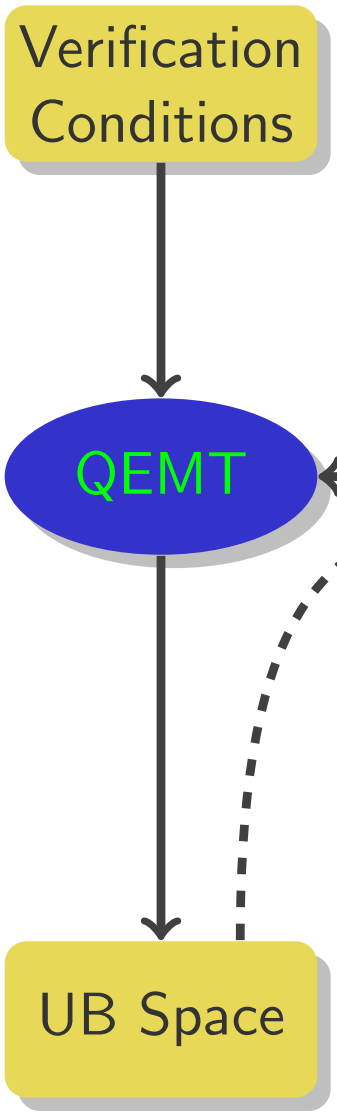
Template Upper Bounds

2. **Eliminate $\forall$ Quantifiers** on program variables

$$\tilde{f}_{main}(t, m, t') = \mu_1 sm + \mu_2 m^2 + \mu_3 mt' + \mu_4 s + \mu_5 m + \mu_6 t' + \mu_0$$

$$\tilde{f}_{rpop}(t, m, t') = \lambda_1 sm + \lambda_2 mt' + \lambda_3 s + \lambda_4 m + \lambda_5 t' + \lambda_0$$

New Approach to Cost Analysis



$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$

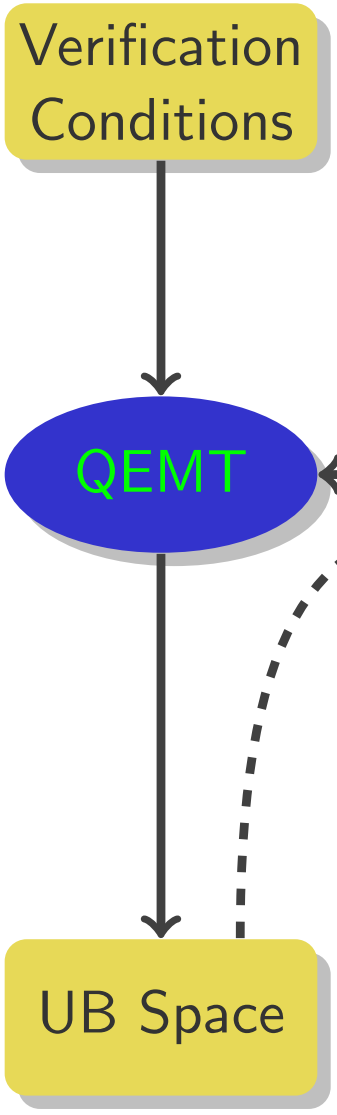
$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

Quantifier Elimination of $\forall \bar{w}.\Phi[\bar{w}, \bar{u}]$ gives a constraint $\psi[\bar{u}]$ on template parameters

$$\tilde{f}_{main}(t, m, t') = \mu_1 sm + \mu_2 m^2 + \mu_3 mt' + \mu_4 s + \mu_5 m + \mu_6 t' + \mu_0$$

$$\tilde{f}_{rpop}(t, m, t') = \lambda_1 sm + \lambda_2 mt' + \lambda_3 s + \lambda_4 m + \lambda_5 t' + \lambda_0$$

New Approach to Cost Analysis



$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$

$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

$$\tilde{f}_{main}(t, m, t') = \mu_1 sm + \mu_2 m^2 + \mu_3 mt' + \mu_4 s + \mu_5 m + \mu_6 t' + \mu_0$$

$$\tilde{f}_{rpop}(t, m, t') = \lambda_1 sm + \lambda_2 mt' + \lambda_3 s + \lambda_4 m + \lambda_5 t' + \lambda_0$$

$$\mu_0 \geq 0 \quad \mu_4 = \lambda_1 - \lambda_5 \quad \mu_5 + \mu_2 \geq \mu_4 + \lambda_0 + \lambda_4 + 1$$

$$\mu_3 = 0 \quad \mu_6 \geq \lambda_5 - \lambda_1 \quad 2 \cdot \mu_2 \geq \lambda_1 + \lambda_4$$

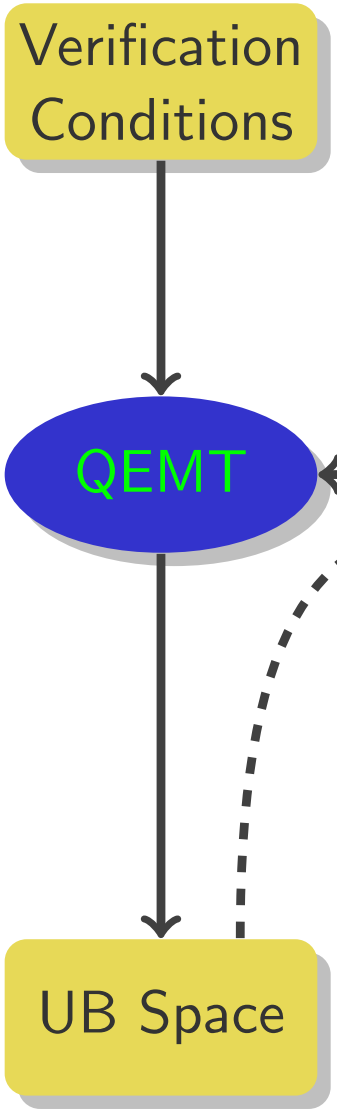
$$\mu_1 = \lambda_1$$

$$\lambda_4 \geq 0 \quad \lambda_0 + \lambda_4 \geq 0$$

$$\lambda_1 \geq 1 \quad \lambda_1 \geq \lambda_3 + \lambda_5 \quad \lambda_1 + \lambda_3 \geq 1$$

$$\lambda_2 = -\lambda_1 \quad \lambda_3 + \lambda_5 \geq 0$$

New Approach to Cost Analysis



$$\forall * .\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$
$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{main}(t, m, t_1) \geq \tilde{f}_{rpop}(t, m, t') + 1 + \tilde{f}_{main}(t' + 1, m - 1, t_1)$$

$$\forall * .\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$$
$$\forall * .\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t - 1, m, t')$$

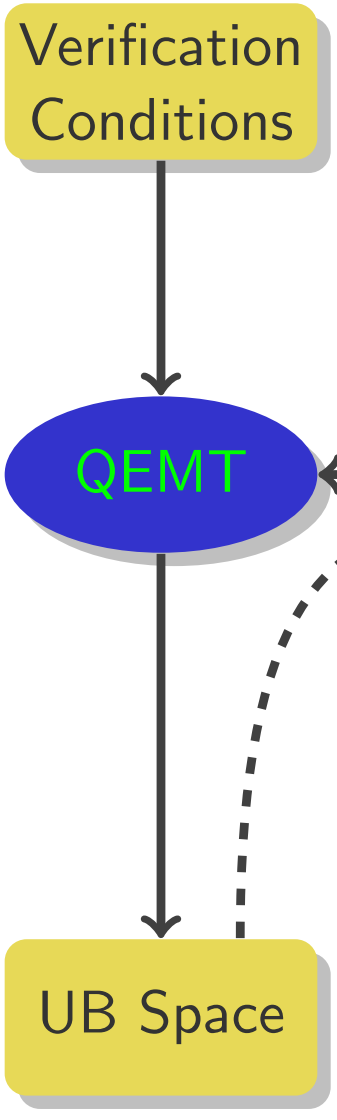
3. Solve the template constraint to get a particular instance

$$\tilde{f}_{main}(t, m, t') = \mu_1 sm + \mu_2 m^2 + \mu_3 mt' + \mu_4 s + \mu_5 m + \mu_6 t' + \mu_0$$
$$\tilde{f}_{rpop}(t, m, t') = \lambda_1 sm + \lambda_2 mt' + \lambda_3 s + \lambda_4 m + \lambda_5 t' + \lambda_0$$

$$\mu_0 \geq 0 \quad \mu_4 = \lambda_1 - \lambda_5 \quad \mu_5 + \mu_2 \geq \mu_4 + \lambda_0 + \lambda_4 + 1$$
$$\mu_3 = 0 \quad \mu_6 \geq \lambda_5 - \lambda_1 \quad 2 \cdot \mu_2 \geq \lambda_1 + \lambda_4$$
$$\mu_1 = \lambda_1$$

$$\lambda_4 \geq 0 \quad \lambda_0 + \lambda_4 \geq 0$$
$$\lambda_1 \geq 1 \quad \lambda_1 \geq \lambda_3 + \lambda_5 \quad \lambda_1 + \lambda_3 \geq 1$$
$$\lambda_2 = -\lambda_1 \quad \lambda_3 + \lambda_5 \geq 0$$

New Approach to Cost Analysis



$\forall *.\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$

$$\mu_1 = 1 \quad \mu_2 = \frac{1}{2} \quad \mu_3 = 0 \quad \mu_4 = 0 \quad \mu_5 = \frac{1}{2} \quad \mu_6 = 0 \quad \mu_0 = 0$$
$$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = 0 \quad \lambda_4 = 0 \quad \lambda_5 = 0 \quad \lambda_0 = 0$$

$\forall *.\ t \geq 0 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t) \geq 0$

$\forall *.\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t-1, m, t')$

Template Upper Bounds

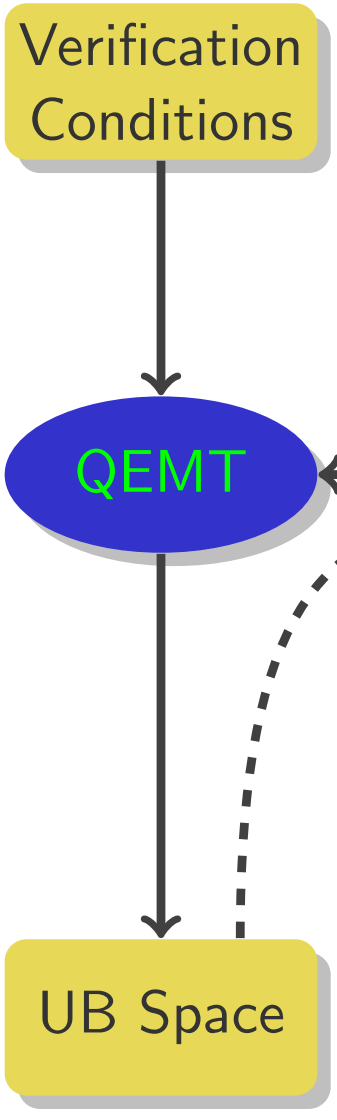
3. Solve the template constraint to get a particular instance

$$\tilde{f}_{main}(t, m, t') = \mu_1 sm + \mu_2 m^2 + \mu_3 mt' + \mu_4 s + \mu_5 m + \mu_6 t' + \mu_0$$
$$\tilde{f}_{rpop}(t, m, t') = \lambda_1 sm + \lambda_2 mt' + \lambda_3 s + \lambda_4 m + \lambda_5 t' + \lambda_0$$

$$\mu_0 \geq 0 \quad \mu_4 = \lambda_1 - \lambda_5 \quad \mu_5 + \mu_2 \geq \mu_4 + \lambda_0 + \lambda_4 + 1$$
$$\mu_3 = 0 \quad \mu_6 \geq \lambda_5 - \lambda_1 \quad 2 \cdot \mu_2 \geq \lambda_1 + \lambda_4$$
$$\mu_1 = \lambda_1$$

$$\lambda_4 \geq 0 \quad \lambda_0 + \lambda_4 \geq 0$$
$$\lambda_1 \geq 1 \quad \lambda_1 \geq \lambda_3 + \lambda_5 \quad \lambda_1 + \lambda_3 \geq 1$$
$$\lambda_2 = -\lambda_1 \quad \lambda_3 + \lambda_5 \geq 0$$

New Approach to Cost Analysis



Template Upper Bounds

$$\forall *.\ t \geq 0 \wedge m = 0 \Rightarrow \tilde{f}_{main}(t, m, t) \geq 0$$

$$\mu_1 = 1 \quad \mu_2 = \frac{1}{2} \quad \mu_3 = 0 \quad \mu_4 = 0 \quad \mu_5 = \frac{1}{2} \quad \mu_6 = 0 \quad \mu_0 = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = 0 \quad \lambda_4 = 0 \quad \lambda_5 = 0 \quad \lambda_0 = 0$$

$$\forall *.\ t \geq 0 \wedge m \geq 1 \Rightarrow f_{rpop}(t, m, t) \geq 0$$

$$\forall *.\ t \geq 1 \wedge m \geq 1 \Rightarrow \tilde{f}_{rpop}(t, m, t') \geq m + \tilde{f}_{rpop}(t-1, m, t')$$

3. Solve the template constraint to get a particular instance

$$\tilde{f}_{main}(t, m, t') = \mu_1 sm + \mu_2 m^2 + \mu_3 mt' + \mu_4 s + \mu_5 m + \mu_6 t' + \mu_0$$

$$\tilde{f}_{rpop}(t, m, t') = \lambda_1 sm + \lambda_2 mt' + \lambda_3 s + \lambda_4 m + \lambda_5 t' + \lambda_0$$

$$\mu_0 \geq 0 \quad \mu_4 = \lambda_1 - \lambda_5 \quad \mu_5 + \mu_2 \geq \mu_4 + \lambda_0 + \lambda_4 + 1$$

$$\mu_3 = 0$$

$$\mu_1 = \lambda$$

$$\lambda_4 \geq 0$$

$$\lambda_1 \geq 1$$

$$\lambda_2 = -\lambda_1$$

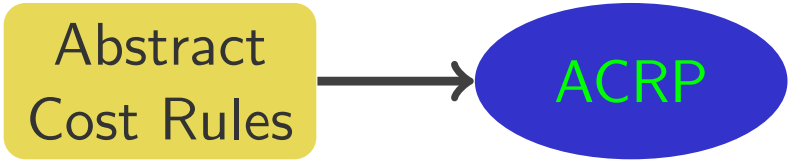
$$\tilde{f}_{main}(t, m, t') = t * m + \frac{m^2+m}{2}$$

$$\tilde{f}_{rpop}(t, m, t') = t * m - t' * m$$

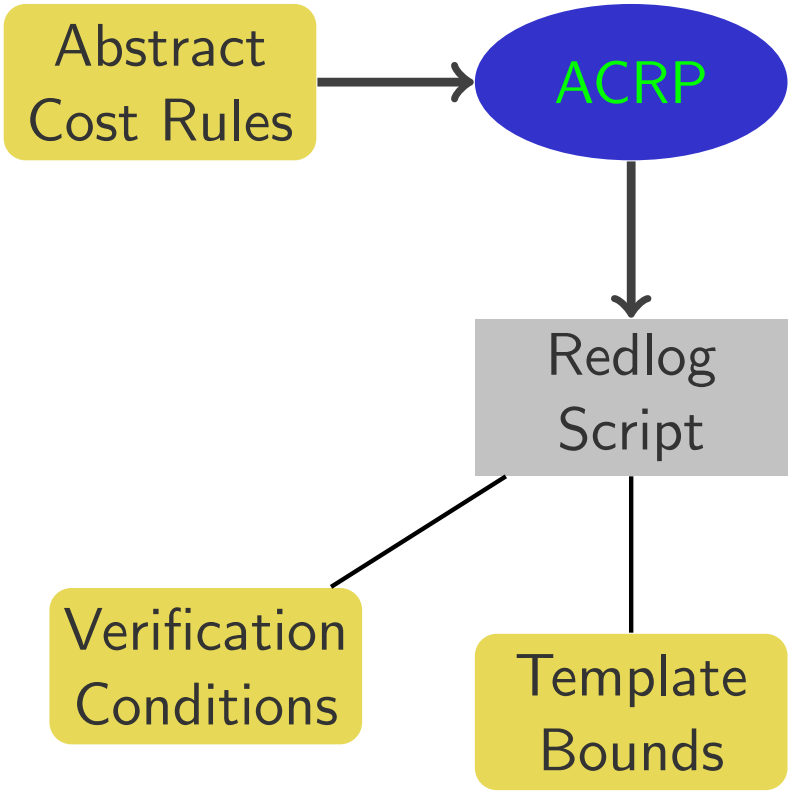
Our prototype

Abstract
Cost Rules

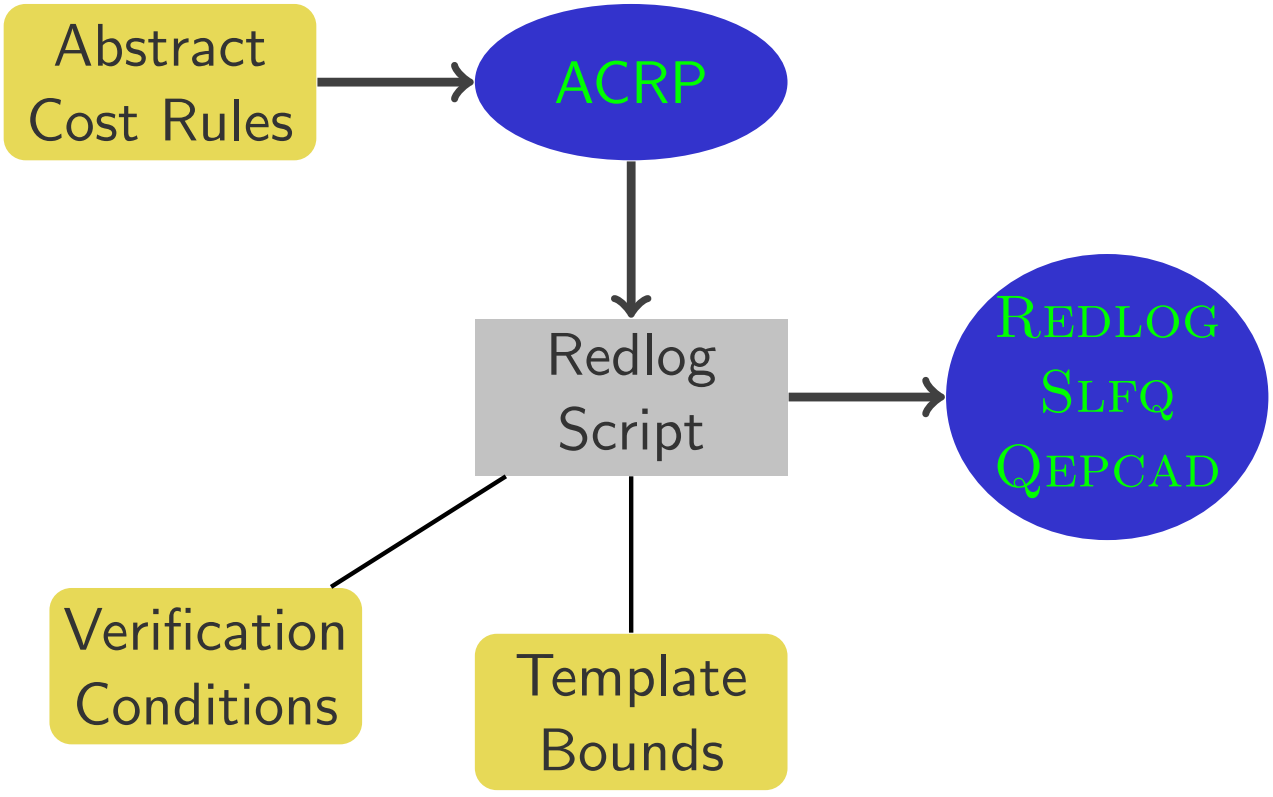
Our prototype



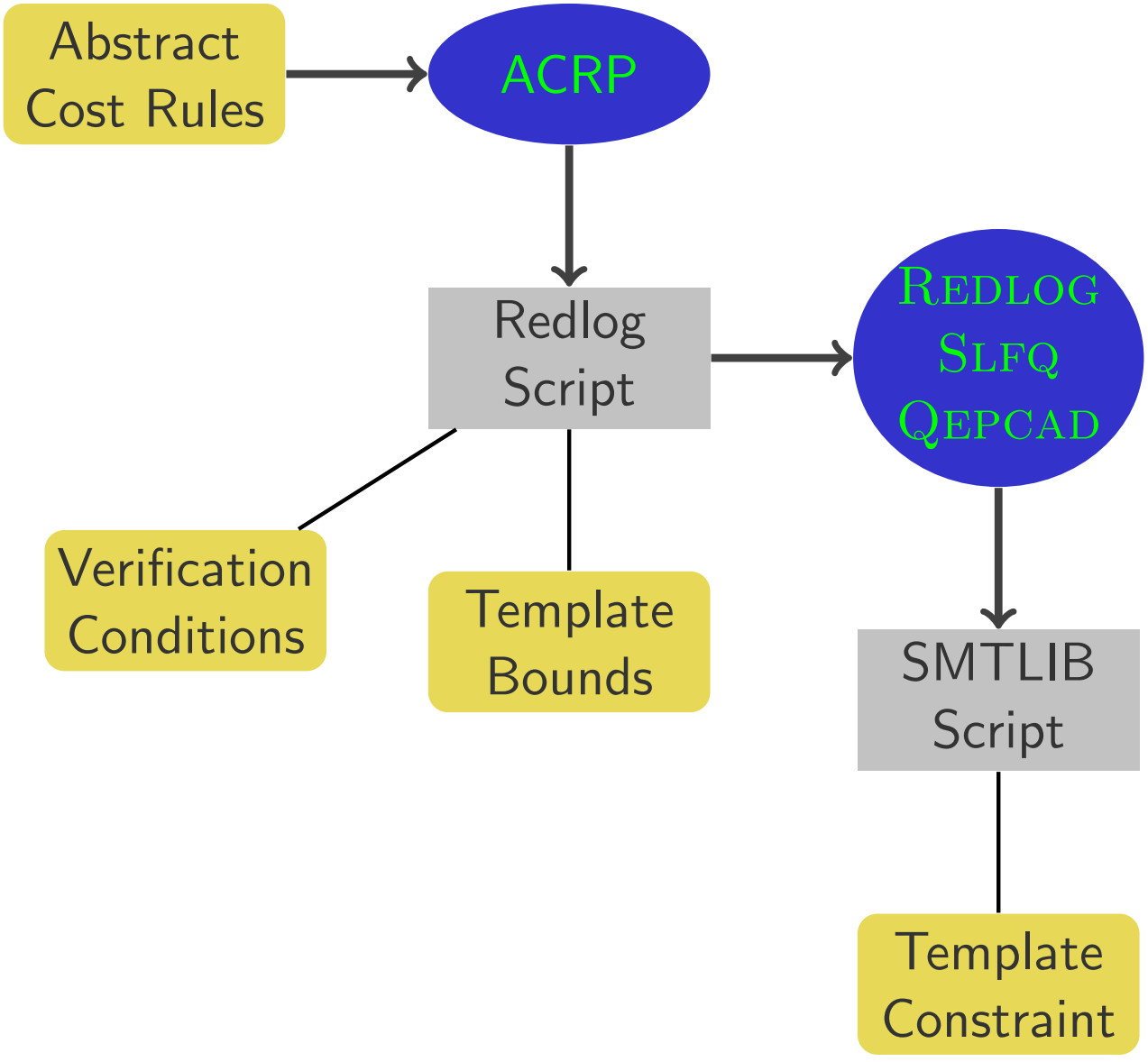
Our prototype



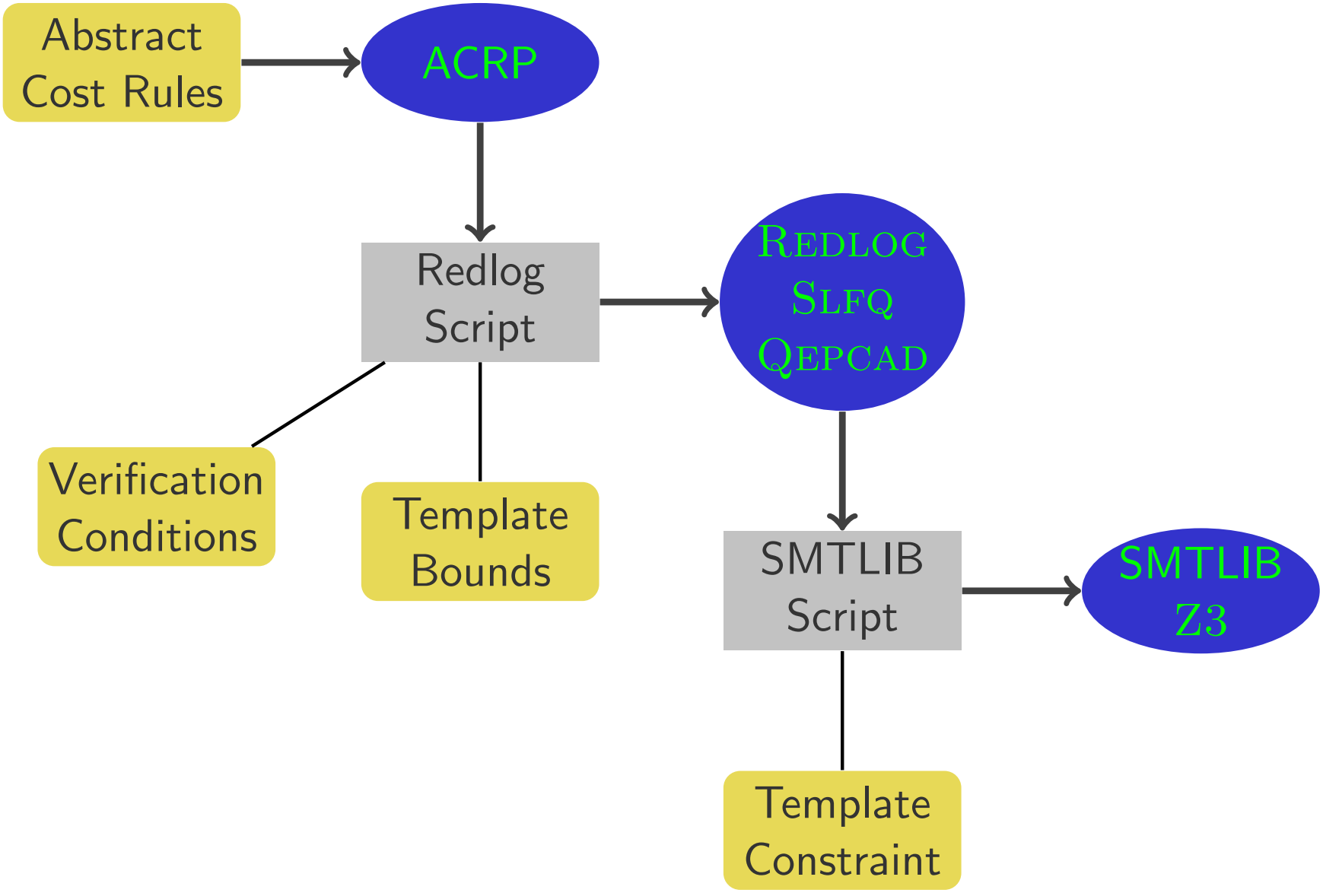
Our prototype



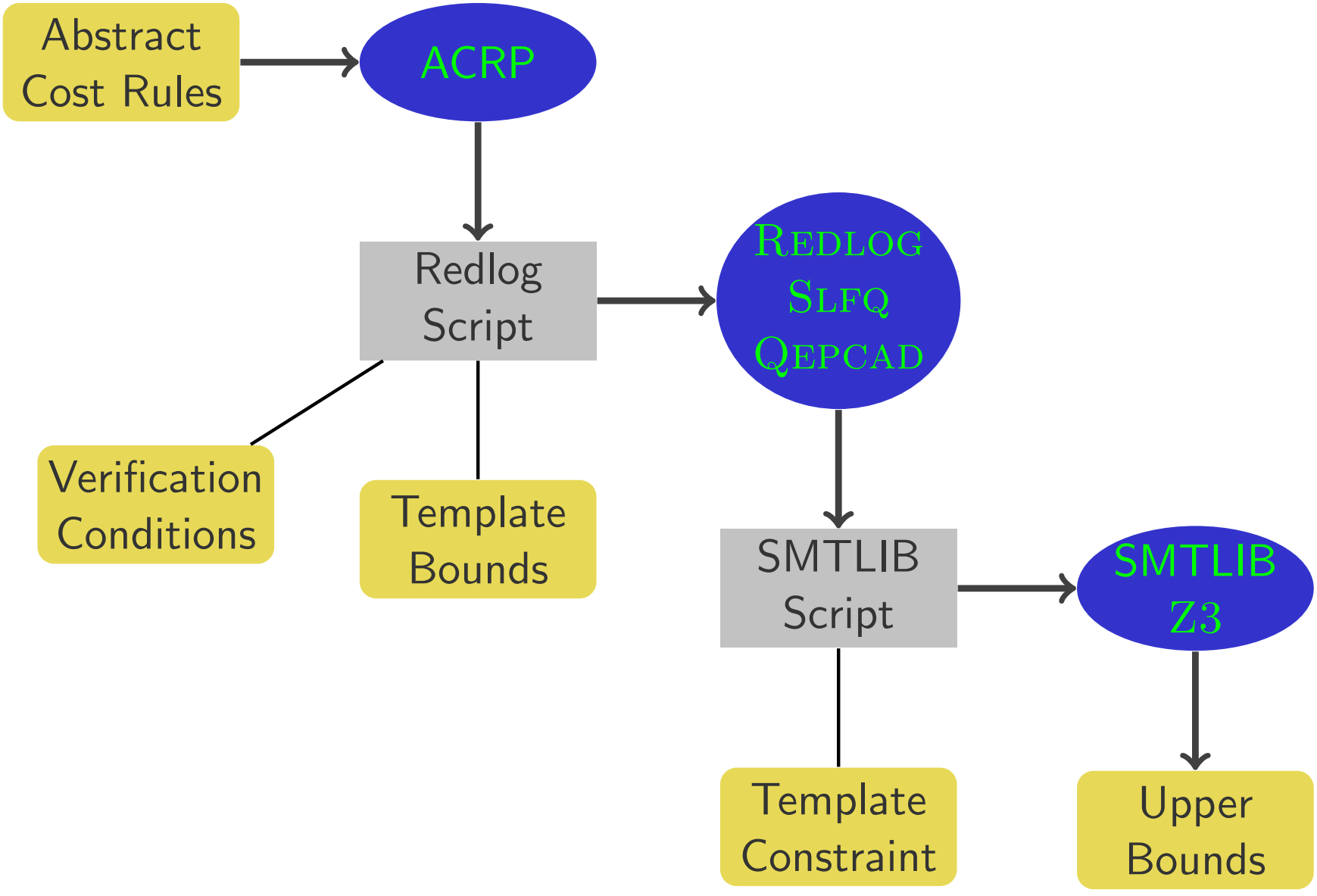
Our prototype



Our prototype



Our prototype



Conclusions

We wanted to understand the differences between different approaches to cost analysis: Classical and Amortised

- Some programs show an Output-Cost codependence
- The amortised approach to cost analysis handles them with precission
- The classical approach ignores these Out-Cost dependencies
- and is unable to get precise bounds for these examples

Our solution:

- **Net-Cost** upper bounds, that take both inputs and outputs
- An approach to *verification* of net-cost Upper Bounds
- its extension to template-based *synthesis*
- Extension to Peak cost of noncumulative resources
- Works also for synthesizing Lower Bounds